

DISPERSION FOR THE WAVE EQUATION INSIDE STRICTLY CONVEX DOMAINS II: THE GENERAL CASE

OANA IVANOVICI, RICHARD LASCAR, GILLES LEBEAU, AND FABRICE PLANCHON

ABSTRACT. We consider the wave equation on a strictly convex domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with smooth boundary $\partial\Omega \neq \emptyset$, with Dirichlet boundary conditions. We construct a sharp local in time parametrix and then proceed to obtain dispersion estimates: our fixed time decay rate for the Green function exhibits a $t^{1/4}$ loss with respect to the boundary less case. Moreover, we precisely describe where and when these losses occur and relate them to swallowtail type singularities in the wave front set, proving that the resulting decay is optimal.

1. INTRODUCTION

Let us consider the wave equation on a manifold (Ω, g) , with a strictly convex boundary $\partial\Omega$ (a precise definition of strict convexity will be provided later):

$$(1.1) \quad \begin{cases} (\partial_t^2 - \Delta_g)u = 0, & \text{in } \Omega \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Δ_g denotes the Laplace-Beltrami operator on Ω .

On any smooth Riemannian manifold without boundary, one may construct an approximate solution, e.g. a parametrix, to any order by microlocal methods. In a suitable patch around $x_0 \in \Omega$ (within the radius of injectivity at x_0), such an approximate solution will be a Fourier integral operator whose phase is a solution to the eikonal equation. Such a phase will be non degenerate in a suitable way and one recovers the same pointwise decay estimates for the kernel of such parametrix as for the flat wave equation: let us denote by $e^{\pm it\sqrt{-\Delta_g}}$ the half-wave propagators on Ω with $\partial\Omega = \emptyset$, and $\psi \in C_0^\infty(]0, \infty[)$. Then, possibly only for (small) finite $|t|$, we have the so-called dispersion estimate,

$$(1.2) \quad \|\psi(-h^2\Delta_g)e^{\pm it\sqrt{-\Delta_g}}\|_{L^1 \rightarrow L^\infty} \leq C(d)h^{-d} \min\{1, (h/|t|)^{\frac{d-1}{2}}\}.$$

Such fixed time decay estimates have been the key tool to obtain other families of estimates, from Strichartz estimates to spectral projector estimates, all of which are space-time estimates in (mixed) Lebesgue spaces, for data in Sobolev spaces. These in turn are invaluable tools for studying a large range of problems, from nonlinear waves to localization of eigenfunctions.

Date: May 31, 2016.

authors were partially supported by A.N.R. grant GEODISP and ERC project SCAPDE.

In the presence of a boundary, much less is known on the decay of the wave equation. In fact, before our recent work [8] on the wave equation on a model strictly convex domain, there were no known results on fixed time dispersion, even with lesser bounds than (1.2). Boundaries induce reflections, and the geometry of broken light rays can be quite complicated. These already cause difficulties in proving propagation of singularities results, and obtaining such results led to major developpements [1, 11, 13, 14], along with constructions of suitable parametrices, see [3, 16, 17]. However, such parametrices, while efficient at proving that singularities travel along the (generalized) bi-characteristic flow, do not seem strong enough to obtain dispersion, at least in the presence of gliding rays and the more flexible microlocal energy arguments from [11, 13, 14] do not provide any information on the amplitude of the wave. Nevertheless, outside strictly convex obstacles, parametrices from [16, 17] were instrumental in matching results from \mathbb{R}^d : Strichartz estimates for the wave equation were obtained in [18], and dispersion estimates were finally proved to hold for $d = 3$ in very recent work of two of the present authors ([7]). For generic boundaries, some positive results for mixed space-time estimates ([19, 2] and references therein) have been obtained by using the machinery developped for low regularity metrics ([20]): reflecting the metric across the boundary and considering a boundary less manifold with a Lipschitz metric across an interface. These arguments require to work on very short time intervals, in order to consider only one reflection (and this, in turn, induces losses when summing time intervals). In fact, counterexamples to the full set of Strichartz estimates inside a strictly convex domain were later constructed in [5, 6], by carefully propagating a cusp singularity along the boundary and across a large number of successive reflections, and these carefully crafted solutions provided hindsight for the parametrix construction on the model domain from [8].

Before stating our main result, we define what we mean by manifold with a strictly convex boundary: given a manifold (Ω, g) where g is a Riemannian smooth metric, its boundary $\partial\Omega \neq \emptyset$ is said to be strictly (geodesically) convex if the induced second fundamental form on $\partial\Omega$ is positive definite. If Ω is actually a domain in \mathbb{R}^d with the identity metric, this definition is equivalent to strict positivity of all principal curvatures at any point of the boundary, and Ω is a strictly convex domain (it admits a gauge function that is strictly convex).

Theorem 1.1. *Let $d \geq 2$ and Ω be a compact manifold with a strictly convex boundary. Let $\psi \in C_0^\infty(\mathbb{R}_*^+)$. There exists $C > 0$, $T_0 > 0$ and $a_0 > 0$ such that for every $a \in]0, a_0]$, $h \in (0, 1]$ and $t \in [-T_0, T_0]$, the solution u_a to (1.1) with data $(u_0, u_1) = (\delta_a, 0)$, where δ_a is the Dirac mass at distance a from $\partial\Omega$, satisfies*

$$(1.3) \quad |\psi(-h^2 \Delta_g) u_a(t, x)| \leq \frac{C}{h^d} \min \left\{ 1, \left(\frac{h}{|t|} \right)^{\frac{d-2}{2} + \frac{1}{4}} \right\}.$$

Remark 1.2. by finite speed of propagation for the wave equation, estimate (1.3) is really local in time **and** space. As such, the compact assumption on Ω may be dropped if one is willing to either assume an uniform lower bound on the second fundamental form or work in a neighborhood of the boundary where such a bound holds.

The dispersion estimate (1.3) may be compared to (1.2): we notice a $1/4$ loss in the h/t exponent, which we may informally relate to the presence of caustics in arbitrarily small times if a is small. Moreover, one of the key features in Theorem 1.1 is that T_0 depends only on the geometry of $\partial\Omega$ and the metric g : (1.3) holds **uniformly** with respect to both the source point and its distance a to the boundary and the frequency $1/h$. In fact, say for $a = h^\theta$, $\theta > 0$, there are at most $1/\sqrt{a} = h^{-\theta/2}$ reflections, and caustics in between them, as we will see later; so in the large frequencies regime $h \rightarrow 0$, we have to deal with an increasingly large number of caustics, even to travel a small distance over a small time T_0 . These caustics occur because optical rays are no longer diverging from each other in the normal direction, where less dispersion occurs when compared to the \mathbb{R}^d case. In fact, we can track caustics and therefore Theorem 1.1 is optimal.

Theorem 1.3. *Let $d \geq 2$ and u_a be the solution to (1.1) with data $(u_0, u_1) = (\delta_a, 0)$. Let $h \in (0, 1]$ and $a \geq h^{4/7-\varepsilon}$. There exists a constant $C > 0$, a finite sequence $(t_n)_n$, $1 \leq n \leq \min(a^{-1/2}, a^{1/2}h^{-1/3})$ with $t_n \sim 4n\sqrt{a}$ and a finite sequence $(x_n)_n$ with $d(x_n, \partial\Omega) \sim a$, such that*

$$h^{-d}(h/t_n)^{\frac{d-2}{2}}n^{-1/4}a^{\frac{1}{8}}h^{1/4} \sim a^{\frac{1}{4}}h^{-d}(h/t_n)^{\frac{d-2}{2}+\frac{1}{4}} \leq C|\psi(-h^2\Delta_g)u_a(t_n, x_n)|.$$

As a byproduct, we get that even for $t \in]0, T_0]$ with T_0 small, the $1/4$ loss is unavoidable for a comparatively small to T_0 and independent of h . Later this optimal loss will be related to swallowtail type singularities in the wave front set of u_a .

Remark 1.4. There is nothing specific about the cosine part of the wave propagator in Theorem 1.1 and 1.3. In fact, both hold equally true if one replaces $\psi(-h^2\Delta_g)u_a(t, x)$ by the half-wave propagators $\psi(hD_t)e^{\pm it\sqrt{-\Delta_g}}\delta_a$ with $\psi \in C_0^\infty(\mathbb{R}^*)$.

As a consequence of (1.3), one obtains by standard arguments that we will not recall the following set of Strichartz estimates.

Theorem 1.5. *Let $d \geq 2$ and u be a solution of (1.1) on a manifold Ω with strictly convex boundary. Then there exists T such that*

$$\|u\|_{L^q(0,T)L^r(\Omega)} \leq C_T \left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)} \right),$$

for (d, q, r) satisfying

$$\frac{1}{q} \leq \left(\frac{d-1}{2} - \gamma(d) \right) \left(\frac{1}{2} - \frac{1}{r} \right), \text{ with } \gamma(d) = \frac{1}{4}$$

and β is dictated by scaling.

In dimension $d = 2$ the known range of admissible indices for which sharp Strichartz are currently known to hold is in fact slightly larger, see [2] where $\gamma(2) = 1/6$. However, in larger dimensions $d \geq 3$, Theorem 1.5 improves the range of indices for which Strichartz estimates hold, and it does so in a uniform way with respect to dimension, in contrast to [2], where $\gamma(3) = 2/3$ and $\gamma(d) = (d-3)/2$ for $d \geq 4$. The results in [2] however apply to any domain or manifold with non-empty boundary.

In the negative direction, counterexamples from [5, 6] prove that $\gamma(d) \geq 1/12$, for $d = 2, 3, 4$. In another very recent work [10], both positive and negative results for the model case when $d = 2$ are pushed further, yielding $1/10 \leq \gamma(2) < 1/9$. These improvements already use the parametrix construction of the present work, as it covers a larger area close to the boundary and prove to be more flexible, even in the model case. We expect these results to extend beyond the model case for $d = 2$, and provide similar improvements in higher dimensions, and these extensions will be addressed elsewhere, as they require significant new developpements that are out of scope here.

In the present work, we focus mainly on the construction of a sharp parametrix for the wave equation (1.1), providing optimal bounds on the amplitude of the wave, including at a discrete set of caustics of swallowtail type that increases to arbitrarily large numbers when the source gets closer and closer to the boundary. While a natural outcome of this parametrix is optimal dispersion bounds, we believe that such a sharp parametrix will prove useful for a broad range of applications beyond the study of dispersive effects, including sharp quantitative versions of propagation of singularities results that are of importance in control theory.

We conclude this introduction with a brief overview of the content in the next sections.

- The first section is devoted to building our parametrix for the wave propagator, which is the key tool to prove Theorems 1.1 and 1.3. While one may think of [8] as providing guidance, the present construction is significantly different for several reasons we briefly outline: unlike in the model case, we do not have an explicit spectral representation. We therefore need to construct quasi-modes, and for this we rely on the parametrix for the Helmholtz equation from [15] (which in turns relies crucially on [12]). Using the Airy-Poisson formula that we introduced in [9], we then obtain a parametrix, both as a “spectral” sum and its counterpart after Poisson resummation. One obvious benefit from this approach is that the Dirichlet boundary condition is easily verified, unlike in [8]. Moreover, the Poisson sum turns out to coincide with the carefully constructed sum of reflected waves in [8], as each term has essentially the same phase (in the model case). The present construction is therefore a sophisticated version of the method of images, which was our inspiration for the construction of suitably matching incoming and outgoing waves in between consecutive reflections in [8] (in turn drawing upon [5]). An additional benefit is that our parametrix holds for any a and h : as such, we extend the reflected waves construction to the range $h^{2/3-\epsilon} < a < h^{4/7}$ (to be a crucial tool in further improvements alluded to earlier, see [10]).
- The second section deals with dispersion estimates for reflected waves. There the analysis of the oscillatory integrals again follows [8] in spirit but it departs from it on several counts. We can no longer reduce the higher dimensional case to $d = 2$ by rotationnal invariance (e.g., the underlying model case is no longer isotropic). For $a < h^{4/7}$, we need to estimate both the size of each wave and their overlap, which is no longer bounded in this regime. To compensate for this, we observe that after a very large number of reflections, the wave starts to exhibit dispersion along the

tangential variable. We therefore obtain a collection of bounds that are sharper and cover an extended region when compared to [8].

- The third section uses the quasi-modes from the parametrix construction in two different directions: first, we properly define the gallery modes and prove that they are almost orthogonal, and that their decay properties are uniform with respect to their discrete parameter, at least in a range useful for later purposes. To our knowledge, these gallery modes had never been defined in such a uniform way in the general case before now ; second, in the regime $a < h^{2/3-\epsilon}$, we use a mix of dispersion estimates on each gallery mode, the spectral sum, and Poisson resummation on the worst terms to obtain a sharper decay than in [8], thereby proving that the worst decay (with a 1/4 loss) really only happens when $h^{1/3} < a < 1$, whereas a lesser 1/6 loss is seen below $a \sim h^{1/3}$, essentially due to cusp propagating and accumulating.
- Finally, the appendix provides hindsight on how to obtain the key properties (and required uniformity, in a suitable sense) of the generating function associated to the equivalence of glancing hypersurfaces in our setting.

2. A PARAMETRIX CONSTRUCTION

By finite speed of propagation, we may work locally near the boundary and chose geodesic normal coordinates on Ω as (x, y) , with $x > 0$ on Ω , $y \in \mathbb{R}^{d-1}$ such that $\partial\Omega = \{(0, y), y \in \mathbb{R}^{d-1}\}$; we write the local coordinates induced by the product $\Omega \times \mathbb{R}_t$ as (x, y, t) . Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(x, y, t, \xi, \eta, \tau)$ on $T^*(\Omega \times \mathbb{R}_t)$. The corresponding local coordinates on the boundary are (y, t, η, τ) . In this coordinates (and up to conjugaison by a non vanishing smooth factor $k(x, y)$), the Laplacian Δ_g becomes (see [4, III, Appendix C])

$$\Delta = k^{-1} \Delta_g k = \partial_x^2 + R(x, y, \partial_y).$$

Since the boundary is everywhere strictly (geodesically) convex, for every point $(0, y_0) \in \partial\Omega$ there exists $\kappa_0 = (0, y_0, \xi_0, \eta_0) \in T^*\Omega$ where the boundary is micro-locally strictly convex, i.e. such that there exists a bicharacteristic passing through $(0, y_0, \xi_0, \eta_0)$ that intersects $\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. The last condition translates into:

$$\tau^2 = R(0, y_0, \eta_0), \quad \{\xi^2 + R(x, y, \eta), x\}|_{\kappa_0} = 2\xi_0 = 0,$$

$$\{\{\xi^2 + R(x, y, \eta), x\}, \xi^2 + R(x, y, \eta)\}|_{\kappa_0} = 2\partial_x R(0, y_0, \eta_0) > 0,$$

where $\{.,.\}$ denotes the Poisson bracket. We will work near such a glancing point and we can assume (without loss of generality) that $y_0 = 0$, hence $\kappa_0 = (0, 0; 0, \eta_0)$. On the boundary, for $(0, y)$ near $(0, 0)$ we have $\partial\Omega$,

$$(2.1) \quad R_0(y, \partial_y) = R(0, y, \partial_y) = \sum_j \partial_{y_j}^2 + O(|y|^2).$$

We let

$$(2.2) \quad R_1(y, \partial_y) = \partial_x R(0, y, \partial_y) = \sum_{j,k} R_1^{j,k}(y) \partial_{y_j} \partial_{y_k}.$$

then we saw that strict convexity for $\partial\Omega$ is equivalent to ellipticity of R_1 (the symbol of R_1 is a positive definite quadratic form). We now define our model Laplacian

$$\Delta_M = \partial_x^2 + \sum_j \partial_{y_j}^2 + x \sum_{j,k} R_1^{j,k}(0) \partial_{y_j} \partial_{y_k},$$

and set

$$q(\eta) = \sum_{j,k} R_1^{j,k}(0) \eta_j \eta_k \quad \text{and} \quad \tau(\omega, \eta) = \sqrt{|\eta|^2 + \omega q(\eta)}^{\frac{2}{3}}.$$

In the sequel we will use various symbols $p(x, y, \eta, \omega, \sigma)$, defined in a conic neighborhood of the set

$$N_0 = \{x = 0, y = 0, \omega = 0, \sigma = 0, \eta \in \mathbb{R}^{d-1} \setminus \{0\}\},$$

the homogeneity being defined by the action of \mathbb{R}_+^*

$$\lambda(x, y, \eta, \omega, \sigma) = (x, y, \lambda\eta, \lambda^{2/3}\omega, \lambda^{1/3}\sigma).$$

A symbol of degree d will be a smooth function defined in a conic neighborhood of N_0 with asymptotic expansion $p \simeq \sum_{k \geq 0} p_{d-k/3}(x, y, \eta, \omega, \sigma)$, with $p_{d-k/3}$ homogeneous of degree $d - k/3$.

The construction of a parametrix near glancing or gliding rays has a long story, starting with the works of Andersson-Melrose [1] and Eskin [3]. We also refer to the thesis of Zworski published in [21], and to the unpublished book of Melrose and Taylor [15] which is available online. We now state an important theorem for our purposes. To our knowledge, this result is stated (for glancing rays) in [21] and a proof is available in [15].

Theorem 2.1. *[Melrose-Taylor, Zworski] There exist phase functions $\psi(x, y, \eta, \omega)$ and $\zeta(x, y, \eta, \omega)$ and symbols $p_0(x, y, \eta, \omega)$ and $p_1(x, y, \eta, \omega)$ such that*

- ψ is homogeneous of degree 1, ζ is homogeneous of degree 2/3, p_0, p_1 are symbols of degree 0;
- the function $G(x, y, \eta, \omega)$ defined by

$$(2.3) \quad G(x, y, \eta, \omega) = e^{i\psi} (p_0 \text{Ai}(\zeta) + x p_1 |\eta|^{-1/3} \text{Ai}'(\zeta))$$

is a solution to

$$(2.4) \quad -\Delta G = \tau^2 G + O_{C^\infty}(|\eta|^{-\infty}) \quad \text{and} \quad \zeta = -\omega + x |\eta|^{2/3} e_0(x, y, \eta, \omega)$$

in a neighborhood of 0 and p_0, e_0 are elliptic in a neighborhood of any $(0, 0, \eta, 0)$ with $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$.

Remark 2.1. Let us recall that the construction of an asymptotic solution of the equation (2.4) of the form (2.3) is a classical result in geometric optics. What is not at all obvious and is the heart of the statement is that such a solution can be construct such that $\zeta|_{x=0} = -\omega$ is independent of (y, η) , and that the term in front of the derivative of Airy in (2.3) vanishes on the boundary $\{x = 0\}$.

Let us now briefly indicates how one can prove theorem 2.1. The first thing to observe is that one can use the Melrose's classification theorem for glancing hypersurfaces (see [12]), in the non-homogeneous setting, " semi-globally" near the glancing set

$$\Sigma_0 = \{(x, y, \xi, \eta), x = 0, y = 0, \xi = 0, |\eta| = 1\}.$$

This is just because the Hamiltonian flow is transversal to Σ_0 .

Therefore, there exists a canonical transform χ_M such that near Σ_0

$$(2.5) \quad \chi_M(\{x = 0, \xi^2 + |\eta|^2 + xq(\eta) = 1\}) = \{X = 0, \Xi^2 + R(X, Y, \Theta) = 1\},$$

The following lemma will be essential for us.

Lemma 2.2. *The generating function for χ_M satisfying (2.5) can be chosen of the form*

$$(X - x)\xi + (Y - y)\eta + \Gamma(X, Y, \xi, \eta),$$

where $\Gamma(0, Y, \xi, \eta)$ is independent of ξ since $\chi_M(\{x = 0\}) = \{X = 0\}$ and can therefore be written under the form

$$(2.6) \quad \Gamma(X, Y, \xi, \eta) = B(Y, \eta) + XA(X, Y, \xi, \eta).$$

The transformation χ_M is then generated by the following relations:

$$(2.7) \quad \begin{cases} x = X + X \frac{\partial A}{\partial \xi}(X, Y, \xi, \eta), \\ y = Y + \frac{\partial B}{\partial \eta}(Y, \eta) + X \frac{\partial A}{\partial \eta}(X, Y, \xi, \eta) \\ \Xi = \xi + A(X, Y, \xi, \eta) + X \frac{\partial A}{\partial X}(X, Y, \xi, \eta) \\ \Theta = \eta + \frac{\partial B}{\partial \eta}(Y, \eta) + X \frac{\partial A}{\partial \eta}(X, Y, \xi, \eta). \end{cases}$$

Moreover, there exists a symbol $p(x, y, \eta, \omega, \sigma)$ of degree 0, elliptic and supported near N_0 , such that

$$(2.8) \quad p|_{x=0} = a(y, \eta, \omega) + (\sigma^2 - \omega - i\partial_\sigma)b$$

where a is a symbol of degree 0 and $b(y, \eta, \omega, \sigma)$ a symbol of degree $-2/3$.

Proof 1. We postponed the proof of this lemma to the appendix, where we will give details on the construction of the generating fonction Γ and the structure of the terms $B(Y, \eta)$ and $A(X, Y, \xi, \eta)$. We just recall that the symmetry property (2.8) on the symbol $p_{x=0}$ is proven in [15].

As an easy byproduct of lemma 2.2, one gets that the function

$$(2.9) \quad G(x, y, \eta, \omega) = \frac{1}{2\pi} \int e^{i(y\eta + \sigma^3/3 + \sigma(xq^{1/3}(\eta) - \omega) + \tau\Gamma(x, y, \sigma q^{1/3}(\eta)/\tau, \eta/\tau))} p d\sigma.$$

satisfies the properties stated in theorem 2.1.

Let $a_0 > 0$ be small and $a \in (0, a_0]$; we denote by $\mathcal{G}(t, x, y, a)$ the Green function for the wave equation

$$(2.10) \quad (\partial_t^2 - \Delta)\mathcal{G} = 0, \text{ for } x > 0, \text{ and } \mathcal{G}|_{x=0} = 0$$

with data $\mathcal{G}|_{t=0} = \delta_{x=a, y=0}$ and $\partial_t \mathcal{G}|_{t=0} = 0$.

Definition 2.3. Let $h \in (0, 1]$ and let $\chi_P(x, y, hD_y)$ be a tangential pseudo-differential operator of degree 0, which equals the identity on Σ_0 and is compactly supported near Σ_0 . We define a parametrix for (2.10) to be a uniform (with respect to a) approximation (modulo $O_{C^\infty}(h^\infty)$) of $\chi_P \mathcal{G}(\cdot, a)$.

We now proceed with such a parametrix construction. We introduce new coordinates $\omega = \frac{\alpha}{h^{2/3}}$, $\eta = \frac{\theta}{h}$, $\sigma = \frac{s}{h^{1/3}}$ in the integral (2.9) defining G and set $\rho(\alpha, \theta) = \sqrt{\theta^2 + \alpha q^{2/3}(\theta)}$ such that $\tau(\omega, \eta) = \frac{\rho(\alpha, \theta)}{h}$. We let

$$(2.11) \quad \Phi(x, y, \theta, \alpha, s) = y\theta + s^3/3 + s(xq^{1/3}(\theta) - \alpha) \\ + \rho(\alpha, \theta)\Gamma(x, y, sq^{1/3}(\theta)/\rho(\alpha, \theta), \theta/\rho(\alpha, \theta)),$$

and

$$p_h(x, y, \theta, \alpha, s) := h^{-1/3}p(x, y, \theta/h, \alpha/h^{2/3}, s/h^{1/3}).$$

We define

$$(2.12) \quad J(f)(x, y) = \frac{1}{2\pi} \int e^{\frac{i}{h}(\Phi(x, y, \theta, \alpha, s) - y'\theta - \varrho\alpha)} p_h f(y', \varrho) dy' d\varrho d\theta d\alpha ds.$$

Lemma 2.4. *The operator J defined above is a semi-classical Fourier Integral Operator associated to a canonical transform χ_J such that*

$$\chi_J(y' = 0, \varrho = 0, |\theta| = 1, \alpha = 0) = \{y = 0, x = 0, |\theta| = 1, \xi = 0\}.$$

Moreover, J is elliptic on the above set and we have

$$-h^2 \Delta J(f) = J(\rho^2(hD_\varrho, hD_{y'})f) \text{ mod } O_{C^\infty}(h^\infty).$$

The Lemma follows from Theorem 2.1 (as p_0 is elliptic and p_1 vanishes on $\partial\Omega$).

Remark 2.5. When $\Gamma = 0$ (the model case), this canonical transform is given explicitly.

Remark 2.6. For technical reasons, it will be convenient to replace the phase function $\psi(x, y, \eta, \omega)$ introduced in Theorem 2.1 by $\psi(x, y, \eta, \omega) - \psi(0, 0, \eta, \omega)$ and the generating function $\Gamma(x, y, u, v)$ that we introduced in Lemma 2.2 by $\Gamma(x, y, u, v) - \Gamma(0, 0, u, v)$. Then we substitute the corresponding phase in our definition of J . This is essentially Kuranishi's trick.

We now digress and present a variation on the Poisson summation formula, which we dub the ‘‘Airy-Poisson summation formula’’. For $z \in \mathbb{C}$ we set

$$A_\pm(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3} z), \text{ then } Ai(-z) = A_+(z) + A_-(z).$$

Lemma 2.7. *Let*

$$L(\omega) = \pi + i \log \frac{A_-(\omega)}{A_+(\omega)}, \quad \text{for } \omega \in \mathbb{R}.$$

Then L is analytic and strictly increasing and we have

$$L(0) = \pi/3, \quad \lim_{\omega \rightarrow -\infty} L(\omega) = 0, \quad L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} - B(\omega^{\frac{3}{2}}), \quad \text{for } \omega \geq 1,$$

with

$$B(u) \simeq \sum_{k=1}^{\infty} b_k u^{-k}, \quad b_k \in \mathbb{R}, \quad b_1 \neq 0.$$

Moreover, the following holds

$$(2.13) \quad Ai(-\omega_k) = 0 \iff L(\omega_k) = 2\pi k \text{ and } L'(\omega_k) = \int_0^\infty Ai^2(x - \omega_k) dx.$$

The first part of the Lemma follows easily using the properties of A_\pm and their asymptotic behavior; the statement (2.13) follows by explicit computations.

Lemma 2.8. *In $\mathcal{D}'(\mathbb{R}_\omega)$, one has*

$$(2.14) \quad \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \delta(\omega - \omega_k).$$

The Lemma is easily proved using the usual Poisson summation formula for e^{-iNx} followed by the change of variable $x = L(\omega)$, which allows to index the sum on the right hand side using the zeros of the Airy function instead of integers. \square

Let us now define, for $\omega \in \mathbb{R}$ (and where we recall that $\alpha = h^{2/3}\omega$)

$$(2.15) \quad K_\omega(f)(t, x, y) = \frac{h^{2/3}}{2\pi} \int e^{\frac{i}{h}(t\rho(h^{2/3}\omega, \theta) + \Phi(x, y, \theta, h^{2/3}\omega, s) - y'\theta - \varrho h^{2/3}\omega)} p_h f(y', \varrho) dy' d\varrho d\theta ds.$$

Then we have

$$J(f)(x, y) = \int_{\mathbb{R}} K_\omega(f)(0, x, y) d\omega,$$

and setting again $\eta = \frac{\theta}{h} \in \mathbb{R}^{d-1}$

$$(2.16) \quad K_\omega(f)(t, x, y) = h^{d-1/3} \int e^{it\tau(\omega, \eta)} G(x, y, \eta, \omega) \hat{f}(\eta, \frac{\omega}{h^{1/3}}) d\eta.$$

Lemma 2.9. *There exists a smooth function $g_{h,a}(y', \varrho)$ such that*

$$J(g_{h,a}) = \frac{1}{2} \delta((x = a, y = 0)) + \text{remaining term},$$

where the remaining term is smooth in some neighborhood of Σ_0 .

The lemma follows easily from the aforementioned fact that J is an elliptic F.I.O. \square

Definition 2.10. Define $\mathcal{P}_{h,a}$ by the distributional bracket

$$\mathcal{P}_{h,a}(t, x, y) = \left\langle \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, K_{\omega}(g_{h,a})(t, x, y) \right\rangle_{\omega}$$

and remark that we have, by the Airy-Poisson formula (2.14),

$$(2.17) \quad \mathcal{P}_{h,a}(t, x, y) = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} K_{\omega_k}(g_{h,a})(t, x, y).$$

Our main result in this section is the following proposition:

Proposition 2.11. *The function $\mathcal{P}_{h,a}$ is a parametrix in the sense of the Definition 2.3.*

First, recall that $K_{\omega}(f)$ is given by (2.16) and from (2.3) we get

$$(\partial_t^2 - \Delta)\mathcal{P}_{h,a} = 0 \text{ mod } O_{C^\infty}(h^\infty) \text{ near } (0, 0).$$

On the other hand, from Theorem 2.1 we have

$$G(0, y, \eta, \omega_k) = e^{i(\psi(0, y, \eta, \omega_k) - \psi(0, 0, \eta, \omega_k))} p_0 Ai(-\omega_k),$$

which immediately yields $G(0, y, \eta, \omega_k) = 0$. Therefore, from (2.17) we get

$$\mathcal{P}_{h,a}(t, x, y)|_{\partial\Omega} = 0,$$

in other words, the Dirichlet boundary condition holds for $\mathcal{P}_{h,a}$. We are left to prove that $\mathcal{P}_{h,a}$ is the correct initial value, which turns out to be the most difficult part. Write

$$\mathcal{P}_{h,a}(0, x, y) = \sum_{N \in \mathbb{Z}} V_N(x, y) \text{ with } V_N(x, y) = \int e^{-iNL(\omega)} K_{\omega}(g_{h,a})(0, x, y) d\omega.$$

We now seek to rewrite V_N as an oscillatory integral; we start with $g_{h,a}$.

Lemma 2.12. *There exists a smooth phase function $\psi_a(\varrho, \theta)$ and a symbol \tilde{p}_h of degree $1/3$ such that*

$$g_{h,a}(y, \varrho) = h^{-2d} \int e^{\frac{i}{h}(\psi_a(\varrho, \theta) + y'\theta)} \tilde{p}_h d\theta$$

and

$$\psi_a(\varrho, \theta) = \varrho^3/3 + a(\varrho q^{1/3}(\theta) + O(\varrho^2)) + O(a^2).$$

Moreover, ψ_a is the critical value of the phase function $\varrho\alpha - \Phi(a, 0, \theta, \alpha, s)$ with respect to the critical points α and s .

Proof. Recall that $0 < a \leq a_0$ for some small a_0 . Let $T_{X=a, Y=0}^*$ be the conormal bundle of $\{X = a, Y = 0\}$, and notice that ψ_a generates $\chi_J^{-1}(T_{X=a, Y=0}^*)$. For $N \in \mathbb{Z}$, we define the phase function Ψ_N (corresponding to the N th reflection from $\chi_J^{-1}(T_{X=a, Y=0}^*)$):

$$(2.18) \quad \Psi_N(x, y, y', \theta, \alpha, s, \varrho) = -NhL(\alpha/h^{2/3}) + s^3/3 + s(xq^{1/3}(\theta) - \alpha) \\ + (y - y')\theta - \varrho\alpha + \psi_a(\varrho, \theta) + \rho(\Gamma(x, y, sq^{1/3}/\rho, \theta/\rho) - B(0, \eta)),$$

where we recall that B was obtained in Lemma 2.2 and given by (2.6).

Recall that we may invert the operator J defined by (2.12) to obtain $g_{h,a}$ such that $J(g_{h,a}) = \frac{1}{2}\delta_{(a,0)}$ modulo smooth remaining terms ; therefore, using the simple form $\Phi - y'\theta - \varrho\alpha$ of the phase function of J , there exists a symbol p'_h of degree 1/3 with

$$(2.19) \quad g_{h,a}(y', \varrho) = h^{-2d-1} \int e^{\frac{i}{h}(-\Phi(a,0,\theta,\alpha,s)+y'\theta+\varrho\alpha)} p'_h d\theta d\alpha ds.$$

We apply the stationary phase in the last integral with respect to α and s : let (α_c, s_c) denote the critical points, then we let $\psi_a(\varrho, \theta) = \varrho\alpha_c - \Phi(a, 0, \theta, \alpha_c, s_c)$, where (ϱ, a) are parameters in a neighbourhood of $(0, 0)$.

Remark 2.13. When $a = 0$, $y'\theta + \varrho\alpha - \Phi(a, 0, \theta, \alpha, s) = y'\theta - s^3/3 + \alpha(s + \varrho)$ (indeed, since at $x = 0$ the phase function Γ is independent of its third variable) and the critical points are such that $s + \varrho = 0$, $s^2 = \alpha$ and $\psi_0(\varrho, \theta) = \varrho^3/3$.

We compute

$$\partial_a \psi_a|_{a=0} = -s_c q^{1/3}(\theta) - \rho_c \partial_a \Gamma(0, 0, s_c q^{1/3}/\rho_c, \theta/\rho_c),$$

and we expand Γ given by (2.6) near the glancing set $\{x = 0, y = 0, \xi = 0, |\theta| = 1\}$,

$$B = B_0(y, \omega) + (r - 1)B_2(y, \omega) + (r - 1)^2 B_2^1(y, \omega, r),$$

where B_0 is homogeneous of degree 1, $B_0 = O(y^3)$, $B_2 = O(y^2)$ and where we set $\omega = \theta/|\theta|$, $r = |\theta|$ (see Section 5.1 in the Appendix for details on B and A). We also have

$$A = \xi\ell + \gamma(\xi^2 + \eta^2 - 1) + \Theta_{\geq 3},$$

where $\ell(y, \omega)$ and $\gamma(y, \omega)$ are smooth functions, $\ell(0, \omega) = 0$, $\gamma(0, \omega) = \{R_0, R_1\}(0, \omega)/(6(R_1)^2)$ and $\Theta_{\geq k}$ denotes any function which is an expansion $\sum_{j \geq k} f_j$, with f_j homogeneous of order j with respect to weights on variable (x, y, ξ, η) , $x : 2, r - 1 : 2, \xi : 1, (y, \omega) : 0$. We have

$$\Gamma(a, y, sq^{1/3}/\rho, \theta/\rho) = B(y, \theta/\rho) + a(\xi\ell + \gamma(\xi^2 + \eta^2 - 1) + \Theta_{\geq 3}) = B(y, \theta/\rho) + a(\xi\ell + \Theta_{\geq 2}).$$

As $\Theta_{\geq 2}$ is the sum of $\xi^2 F + aG + (r - 1)H$, we have $\partial_a \Gamma|_{a=0} = O(\varrho^2)$ when $y = 0$ as $\ell(0, \omega) = 0$, $\xi = s_c q^{1/3}/\rho = O(\varrho)$, $|\eta| - 1 = O(s^2) = O(\varrho^2)$ as $s = |\theta| + O(\alpha)$, $r - 1 = O(s^2)$. We also have $\partial_a \psi_a|_{a=0} = \varrho q^{1/3}(\theta) + O(\varrho^2)$ therefore

$$\psi_a(\varrho, \theta) = \varrho^3/3 + a(\varrho q^{1/3}(\theta) + O(\varrho^2)) + O(a^2),$$

and $g_{h,a}$ is a phase integral, as $\nabla_{\alpha,s}(\varrho\alpha - \Phi(a, 0, \theta, \alpha, s))$ is invertible in $(\alpha_c, s_c) = (0, 0)$ when $\varrho = a = 0$. This yields, for some new symbol \tilde{p}_h homogeneous of degree $1/3$,

$$g_{h,a}(y', \varrho) = h^{-2d} \int e^{\frac{i}{h}(\psi_a(\varrho, \theta') + y'\theta')} \tilde{p}_h d\theta' + O_{C^\infty}(h^\infty).$$

This achieves the proof of Lemma 2.12. \square

We now insert $g_{h,a}$ into the formula (2.15) defining K_ω and perform stationary phase to eliminate y' and get $\theta' = \theta$, we obtain

$$\mathcal{P}_{h,a}(t, x, y) = \sum_{N \in \mathbb{Z}} V_N(t, x, y),$$

where

$$V_N(t, x, y) = \frac{1}{2\pi h^{d+1}} \int e^{\frac{i}{h}(t\rho(\alpha, \theta) + \Phi - \varrho\alpha + NhL(\alpha h^{-2/3}) + \psi_a(\varrho, \theta))} p_h \tilde{p}_h d\theta d\alpha ds d\varrho.$$

The symbol $p_h \tilde{p}_h$ of V_N is degree 0 and is given by an asymptotic expansion with small parameter h and main term equal to 1 (indeed, since \tilde{p}_h has been obtained by inverting J , whose symbol is p_h).

It remains to show that at time $t = 0$, we do have $\mathcal{P}_{h,a}(0, x, y) = \frac{1}{2}\delta_{x=a, y=0} + O_{C^\infty}(h^\infty)$. By design, $V_N(0, x, y) = \frac{1}{2}\delta(x = a, y = 0) + O_{C^\infty}(h^\infty)$, therefore we are left to proving that, uniformly in $N \neq 0$, $V_N(0, x, y) \in O_{C^\infty}(h^\infty)$. The function $V_N|_{t=0}$ is an oscillatory integral, with phase Ψ_N defined in (2.18). Set

$$(2.20) \quad \rho(B(y, \theta/\rho) - B(0, \theta/\rho)) = y\mathfrak{X}(y, \theta, \alpha),$$

and using the Taylor expansion of B near the glancing set, we have $\mathfrak{X}(y, \theta, \alpha) \in O(y^2, \alpha y, \alpha^2)$ and we set

$$\mathfrak{U}(y, \theta, \alpha) = \theta + \mathfrak{X}(y, \theta, \alpha).$$

We now perform the following change of variables

$$F_* : (y, \theta, \alpha) \rightarrow (y, \mathfrak{U}(y, \theta, \alpha), \alpha).$$

For small y and α, θ near $|\theta| = 1$, F_* is a bijective map, due to the properties of B which were recalled earlier. Define G_* to be its inverse map. Rewrite

$$V_N(0, x, y) = \int p_h^\# e^{\frac{i}{h}\tilde{\Psi}_N} d\mathfrak{U} d\alpha ds d\varrho,$$

where

$$\begin{aligned} \tilde{\Psi}_N = & -NhL(\alpha h^{-2/3}) + s^3/3 + s(xq^{1/3}(\theta(\mathfrak{U})) - \alpha) + \psi_a(\varrho, \theta(\mathfrak{U})) + y\mathfrak{U} - \varrho\alpha \\ & + \rho x A(x, y, sq^{1/3}/\rho, \theta/\rho). \end{aligned}$$

We now rescale variables,

$$x = aX, y = aY, \varrho = a^{\frac{1}{2}}\Upsilon, \alpha = a\mathcal{A}, s = a^{\frac{1}{2}}S, ,$$

which is consistent with the correct homogeneity. Setting $\tilde{V}_N(X, Y) := V_N(0, x, y)$, we are left with

$$\tilde{V}_N(X, Y) = \int e^{\frac{i}{h}\tilde{\Psi}_N} p_h^\sharp d\mathcal{U} d\mathcal{A} dS d\Upsilon, ,$$

for which we aim at proving that $\tilde{V}_N \in O(h^\infty)$ for $X \geq -c_0$, $|Y| \leq c_0/a$ for $N \neq 0$ and a fixed small $c_0 > 0$. Henceforth, our large parameter will be $\lambda = a^{3/2}/h$ (recall $a \geq h^{2/3-\varepsilon}$) and we are considering oscillatory integrals

$$\int e^{i\lambda\Phi_N} \sigma_h d\Upsilon d\mathcal{A} dS, ,$$

where σ_h is a symbol and $\Phi_N = \Psi_{N,0} + a^{1/2}\Psi_*$. Here $\Psi_{N,0}$ is essentially the (anisotropic) model phase function and Ψ_* should be thought of as a error term. First, we observe that Φ_N is non stationnary in Υ provided that \mathcal{A} is small: indeed,

$$\partial_\Upsilon \Phi_N = a^{1/2}(-\mathcal{A} + \Upsilon^2 + q^{1/3} + a^{1/2}\Upsilon f + af),$$

where f is any C^∞ function in $aX, a^{1/2}S, a^{1/2}\Upsilon, a\mathcal{A}, \dots$. Thus, we get decay in Υ unless we assume $\mathcal{A} \geq \mathcal{A}_0 > 0$: replace $L(\omega)$ by its asymptotics for large positive ω , e.g. $L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2})$. Then we have

$$\Psi_{N,0} = -\frac{4}{3}\mathcal{A}^{3/2}N + S(X(1+\ell)q^{1/3} - \mathcal{A}) + \frac{S^3}{3} - \Upsilon\mathcal{A} + \Upsilon^3/3 + q^{1/3}\Upsilon + \frac{N}{\lambda}B(\mathcal{A}^{3/2}\lambda)$$

and

$$\Psi_* = \Upsilon^2 f + f + X^2 f + XS^2 f + AXf.$$

Stationnary points will now solutions to the following three equations

$$\begin{aligned} -2N\mathcal{A}^{\frac{1}{2}}(1 - \frac{3}{4}B'(\lambda\mathcal{A}^{3/2})) - (S + \Upsilon) + a^{\frac{1}{2}}Xf + a^{\frac{1}{2}}f &= 0 \\ -\mathcal{A} + \Upsilon^2 + q^{\frac{1}{3}} + a^{\frac{1}{2}}\Upsilon f + af &= 0 \\ S^2 - \mathcal{A} + Xq^{\frac{1}{3}}(1+\ell) + 2a^{\frac{1}{2}}\frac{\gamma}{\rho}q^{\frac{2}{3}}SX + XO_{\geq 2} &= 0 \end{aligned}$$

from which we easily conclude that if $|N| \geq 2$, decay will follow from integration by parts in \mathcal{A} .

Hence, we are left with $N = 1$. As $\exp(iB(\lambda\mathcal{A}^{3/2}))$ is a symbol of degree zero, the first equation rewrites

$$(S + \Upsilon) + 2\mathcal{A}^{\frac{1}{2}} = a^{\frac{1}{2}}Xf,$$

the third one yields

$$-S^2 + \mathcal{A} = cX \text{ with } c = q^{\frac{1}{3}}(1 + \ell) + 2a^{\frac{1}{2}} \frac{S}{\rho} q^{\frac{2}{3}} + O_{\geq 2} > 0.$$

Then the second equation rewrites

$$\mathcal{A} = \Upsilon^2 + q_1^{\frac{1}{3}} + a^{\frac{1}{2}} f \Upsilon + af,$$

where $q_1^{1/3} = q^{1/3} + O(y)$. Using the first equation,

$$S^2 - \mathcal{A} = (\Upsilon + 2\mathcal{A}^{\frac{1}{2}} + a^{\frac{1}{2}} c(S^2 - \mathcal{A})f)^2 - \Upsilon^2 - q^{\frac{1}{3}} - a^{\frac{1}{2}} \Upsilon f - af,$$

and therefore

$$c_0(S^2 - \mathcal{A}) = 4\mathcal{A} + 4\mathcal{A}^{\frac{1}{2}} \Upsilon - q^{\frac{1}{3}} + \varepsilon \text{ and } \mathcal{A} = \Upsilon^2 + q^{\frac{1}{3}} + \epsilon,$$

where $c_0 > 0$ and ϵ, ε small. Then, if $|\epsilon|$ is small enough, we have $\Upsilon^2 \leq \mathcal{A}$, hence $4\mathcal{A}^{\frac{1}{2}} \geq 2(\mathcal{A}^{\frac{1}{2}} + |\Upsilon|)$ and as

$$c_0(S^2 - \mathcal{A}) \geq \frac{4\mathcal{A}^{\frac{1}{2}}}{\mathcal{A}^{\frac{1}{2}} + |\Upsilon|} (q^{\frac{1}{3}} - |\epsilon|) - q^{\frac{1}{3}} - |\varepsilon|,$$

we obtain $c_0(S^2 - \mathcal{A}) \geq c_1 > 0$, e.g. $S^2 - \mathcal{A} \geq c_2 > 0$ and $X < -c_3$ (which is what we seek: the main contribution should be below the boundary $X = 0$). Finally, we may obtain decay in $X \geq -c_3$, $c_3 > 0$ by integration by parts (see [8, Lemma 2.24]) which concludes the proof. \square

3. DISPERSION ESTIMATES

We now utilize the parametrix to obtain the following dispersion estimates, restricting to positive times for the sake of simplicity.

Theorem 3.1. *There exists $\varepsilon_0, C_0, \varepsilon, C_\varepsilon, c$ such that for all $|(t, x, y, h, a)| < \varepsilon_0$ and $t > h$, one has*

- for $t \leq c\sqrt{a}$,

$$(3.1) \quad |\mathcal{P}_{h,a}(t, x, y)| \leq C_0 h^{-d} \left(\frac{h}{t} \right)^{\frac{d-1}{2}}.$$

- For $a \geq h^{2/3-\varepsilon}$,

$$(3.2) \quad |\mathcal{P}_{h,a}(t, x, y)| \leq C_0 h^{-d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \left(\left(\frac{h}{t} \right)^{\frac{1}{2}} + C_\varepsilon (\sup(a, x))^{\frac{1}{4}} \left(\frac{h}{t} \right)^{\frac{1}{4}} + h^{\frac{1}{3}} \right).$$

- For $a \leq h^{1/3+\varepsilon}$,

$$(3.3) \quad |\mathcal{P}_{h,a}(t, x, y)| \leq C_0 h^{-d} \left(\frac{h}{t} \right)^{\frac{d-2}{2} + \frac{1}{3}}.$$

Let us make a few comments:

- the first estimate, (3.1), is similar to the (short time) dispersion for a free wave. Here, on the given timescale $t < c\sqrt{a}$, the wave has only at most one reflection and singularities in the expanding wave front haven't appeared yet. It is worth pointing out that (a suitable version of) such dispersion is already proved in [2] ;
- the second estimate, (3.2), will be proved using the parametrix as a sum over reflected waves. The first term is the free dispersion, the second term is due to swallowtail singularities and the third and last term is due to the presence of cusps appearing after each swallowtail singularity, between two consecutive reflections ; notice that here, we use the parametrix construction in an extended region $a \geq h^{2/3-\varepsilon}$, when compared to the previous parametrix construction in [8] ($a \geq h^{4/7-\varepsilon}$), and this requires refinements in decay estimates and in estimating the number of reflected waves that overlap;
- The third estimate, (3.3), will be proved using the parametrix as a sum over quasi-modes, and we postpone its proof to the last section.

We start with a lemma which allows to deal with the parametrix “behind the wave front”.

Lemma 3.2. *There exists c_0 and T_0 such that for $0 \leq t \leq T_0$,*

$$\sup_{x,y,t \in \mathcal{B}} |\mathcal{P}_{h,a}(t,x,y)| \leq Ch^{-d}O((h/t)^\infty),$$

where $\mathcal{B} = \{0 \leq x \leq a, |y| \leq c_0 t, 0 < h \leq t\}$.

The lemma will follow from propagation of singularities theorems: for $t \leq a/C_1$ with a large C_1 , we may apply Hörmander's theorem in the interior of Ω .

Define T_0 to be small, and consider a given $T \in [h, T_0]$. We perform a rescaling of all variables, to define (s, X, Y) as new variables and \hbar a new parameter

$$t = Ts, \quad x = TX, \quad y = TY, \quad \text{as well as } \hbar = h/T.$$

Let $f_T(s, X, Y) = f(Ts, TX, TY)$ for any function f , then

$$(\square f)_T = \frac{1}{T^2} \square_T f_T \quad \text{with} \quad \square_T = -\partial_s^2 + \partial_X^2 + T^2 R(TX, TY, T^{-1} D_Y).$$

Set $b = a/T$, $0 < b \leq c_1$ and $\mathcal{P}_{b,T,\hbar} = T^d(\mathcal{P}_{a,h})_T$ one may apply the Melrose-Sjöstrand theorem to $\mathcal{P}_{b,T,\hbar}$ to obtain $\mathcal{P}_{b,T,\hbar} \in O(\hbar^\infty)$ for $0 \leq X \leq b$, $|Y| \leq c_0$ and $s = 1$, when $T \leq T_0$ small enough. In fact, $\mathcal{P}_{b,T,\hbar}$ is a parametrix for \square_T , and \square_T is smooth in T and for T small enough, \square_T is close to the usual wave operator \square_0 , e.g. $\square_0 = -\partial_s^2 + \partial_X^2 + \Delta_Y$ (recall that we picked geodesic normal coordinates at the boundary, and $R_0(y, \eta) = |\eta|^2 + O(|y|^2)$).

If we denote by $\mathcal{G}_{b,T}$ the Green function of \square_T , that is to say the solution to

$$\square_T v = 0$$

in Ω , with $v = 0$ on $\partial\Omega$, $v(t = 0) = \delta_{(b,0)}$ and $\partial_t v(t = 0) = 0$, then we have $\mathcal{P}_{b,T,\hbar} = \chi_P(TX, TY, \hbar D_Y) \mathcal{G}_{b,T}$ and its wave front set is described by the Melrose-Sjöstrand theorem.

We therefore have to check the following about an optical ray $\sigma \in [0, 1] \rightarrow \rho_T(\sigma)$: if it is starting at $\sigma = 0$ from $(b, 0, 0, \xi_0, \eta_0, 1)$ with $\xi_0^2 + R(Tb, 0; \eta_0) = 1$ and $|\eta_0| \geq c_1 > 0$, then, denoting $Y(\sigma) = \pi_Y(\rho_T(\sigma))$, we have $|Y(1)| \geq c_0 > 0$.

Denote by $\eta(\sigma) = \pi_\eta(\rho_T(\sigma))$, then $(Y, \eta)(\sigma)$ are solutions to the Hamilton-Jacobi equations

$$\partial_\sigma Y = \partial_\eta R_T \text{ and } \partial_\sigma \eta = -\partial_Y R_T,$$

with $Y(0) = b$, $\eta(0) = \eta_0$, $R_T(X, Y, \eta) = T^2 R(TX, TY, \eta/T)$. If $c_0 > 0$ is small, then $|Y(\sigma)| \geq 2c_0\sigma - CT\sigma$ for all $\sigma \in [0, 1]$, and if T_0 is small enough, we will get the lower bound $|Y(1)| \geq c_0 > 0$. \square

We now return to our parametrix $\mathcal{P}_{h,a}$: recall $\mathcal{P}_{h,a}$ is given by a sum of phase integrals with $(d+2)$ parameters $\beta = (\theta, \alpha, s, \varrho)$, where $\theta \in \mathbb{R}^{d-1}$ is close to the sphere $|\theta| = 1$ and $\alpha, s, \varrho \in \mathbb{R}$ are in a neighborhood of 0. We have

$$\mathcal{P}_{h,a}(t, x, y) = \sum_{N \in \mathbb{Z}} V_N(t, x, y)$$

where

$$V_N(t, x, y) = h^{-d-1} \int e^{\frac{i}{h} \Psi_N} q_h(t, x, y; \beta) d\theta d\alpha ds d\varrho,$$

and the phase Ψ_N is defined by

$$\begin{aligned} \Psi_N = y\theta + t\rho(\alpha, \theta) - NhL(\alpha h^{-2/3}) + s^3/3 + s(xq^{1/3}(\theta) - \alpha) + \psi_a(\varrho, \theta) - \varrho\alpha \\ + \rho(\alpha, \theta)(\Gamma(x, y, sq^{1/3}/\rho, \theta/\rho) - B(0; \theta/\rho)), \end{aligned}$$

where the symbol q_h is of degree zero in h (recall Γ comes from the generating function of the Melrose transformation).

If we rewrite, like we did for $t = 0$,

$$\rho(B(y, \theta/\rho) - B(0, \theta/\rho)) = y\mathfrak{X}(y, \theta, \alpha),$$

we have $\mathfrak{X}(y, \theta, \alpha) \in O(y^2, \alpha y, \alpha^2)$ for (y, α) close to zero and θ close to the sphere, exactly like after (2.20) (see also the Appendix).

For small y, α , $|\theta| \sim 1$, the map $F_*: (y, \alpha, \theta) \rightarrow (y, \alpha, \Theta(y, \alpha, \theta))$ with $\Theta(y, \alpha, \theta) = \theta + \mathfrak{X}(y, \theta, \alpha)$ is a diffeomorphism; denote $F_*^{-1} = G_*$ its inverse: $(y, \alpha, \Theta) \rightarrow (y, \alpha, \theta(y, \alpha, \Theta))$. If we perform the corresponding change of variables in the V_N integrals, we get a new phase, denoted again Ψ_N ,

$$\begin{aligned} \Psi_N = y\Theta + t\rho(\theta(y, \alpha, \Theta), \alpha) - hNL(\alpha h^{-2/3}) + s^3/3 + s(xq^{1/3}(\theta(y, \alpha, \Theta)) - \alpha) \\ + \psi_a(\varrho, \theta(y, \alpha, \Theta)) - \varrho\alpha + \rho x A(x, y, sq^{1/3}/\rho, \theta/\rho). \end{aligned}$$

Recalling that

$$(3.4) \quad \psi_a(\varrho, \theta) = \varrho^3/3 + a(\varrho q^{1/3}(\theta) + O(\varrho^2)) + O(a^2),$$

we get

$$(3.5) \quad \Psi_N = y\Theta + t|\theta|_{\alpha=0} + \Psi_{N,0} + \alpha\varepsilon + \Psi_*,$$

where

$$\begin{aligned} \Psi_{N,0} = & s^3/3 + s(xq^{1/3}(\theta|_{\alpha=0})(1+\ell)(y, \theta|_{\alpha=0}/|\theta|_{\alpha=0}) - \alpha) + \varrho^3/3 + a\varrho q^{1/3}(\theta|_{\alpha=0}) \\ & - \varrho\alpha - \frac{4}{3}N\alpha^{3/2} + NhB(\alpha^{3/2}/h) + t\alpha \frac{q^{2/3}(\theta|_{\alpha=0})}{2|\theta|_{\alpha=0}}. \end{aligned}$$

This phase $\Psi_{N,0}$ is the phase function of the anisotropic model case (except for the extra factor ℓ). The two other terms in (3.5) should be viewed as error terms arising from $\Gamma(x, y; u, v)$: we have $\varepsilon = \varepsilon_0 t + \varepsilon_1 s x + \varepsilon_2 \varrho a$ with

$$\varepsilon_0 = \left\{ \frac{\partial \theta}{\partial \alpha} \frac{\theta}{|\theta|} \right\}_{|\alpha=0}, \quad \varepsilon_1 = \left\{ \frac{\partial \theta}{\partial \alpha} \right\}_{|\alpha=0} \nabla q^{1/3} (1 + \ell), \quad \varepsilon_2 = \left\{ \frac{\partial \theta}{\partial \alpha} \right\}_{|\alpha=0} \nabla q^{1/3} (\theta|_{\alpha=0})$$

and from $\frac{\partial \theta}{\partial \alpha}|_{\alpha=0} = O(y)$, we get $\varepsilon_j = O(y)$ for $j = 1, 2, 3$. On the other hand,

$$\Psi_* = xs^2 f_1 + x\alpha f_2 + sx^2 f_3 + a\varrho^2 f_4 + a^2 f_5 + t\alpha^2 f_6 + a\varrho\alpha^2 f_7,$$

where for $k = 1, 2, 3, 4, 5, 6, 7$, f_k is a smooth function near $|\theta| = 1$, $\alpha = s = \varrho = 0$.

The identity (3.5) is a straightforward computation in view of the Taylor expansion of Γ near the glancing set as well as (3.4).

In the model case, one has $\Psi_N^{mod} = y\theta + t\rho(\alpha, \theta) + \Psi_{N,0}^{mod}$, where

$$\Psi_{N,0}^{mod} = \frac{s^3}{3} + s(xq^{1/3}(\theta) - \alpha) + \frac{\varrho^3}{3} + a\varrho q^{1/3}(\theta) - \varrho\alpha - \frac{4}{3}N\alpha^{3/2} + NhB(\alpha^{3/2}/h).$$

Both phase functions Ψ_N and Ψ_N^{mod} define Lagrangian manifolds Λ_N and Λ_N^{mod} . If π is the canonical projection on space-time \mathbb{R}^{d+1} , we would like to parametrize $\pi(\Lambda_N)$. We first parametrize $\pi(\Lambda_N^{mod})$: we have the following equations describing Λ_N^{mod}

$$\begin{aligned} t(\theta^2 + \alpha q^{2/3})^{-1/2} q^{2/3} - (s + \varrho) - 2N\alpha^{1/2} + \frac{3}{2}N\alpha^{1/2} B'(\alpha^{3/2}/h) &= 0 \\ s^2 + xq^{1/3} - \alpha &= 0 \\ \varrho^2 + aq^{1/3} - \alpha &= 0 \\ y + t(\theta^2 + \alpha q^{2/3})^{-1/2} \theta + \frac{1}{3q^{2/3}}(t\alpha q^{1/3}(\theta^2 + \alpha q^{2/3})^{-1/2} + sx + a\varrho) \nabla q &= 0. \end{aligned}$$

We now perform a rescaling, with small $a > 0$: $x = aX$, $s = a^{1/2}S$, $t = a^{1/2}T$ and $\varrho = a^{1/2}\Upsilon$. Then $\pi(\Lambda_N^{mod})$ is described by the equations

$$\begin{aligned} X &= (A - S^2)q^{-1/3} \\ -y &= a^{1/2}T(\theta^2 + \alpha q^{2/3})^{-1/2}\theta + \frac{a^{3/2}}{3q^{2/3}}(TAq^{1/3}(\theta^2 + \alpha q^{2/3})^{-1/2} + SX + \Upsilon)\nabla q \\ T &= 2(\theta^2 + \alpha q^{2/3})^{1/2}q^{-2/3}(S + \Upsilon + 2NA^{1/2}(1 - \frac{3}{4}B'(\alpha^{3/2}/h))) \end{aligned}$$

with $A = \Upsilon^2 + q^{2/3}(\theta)$. Rescaling again, $S = q^{1/6}\mathcal{S}$, $\Upsilon = q^{1/6}\mathcal{T}$, $A = q^{1/3}\mathcal{A}$, one has

$$\begin{aligned} X &= \mathcal{A} - \mathcal{S}^2 \\ -y &= a^{1/2}T(\theta^2 + aq\mathcal{A})^{-1/2}\theta + \frac{a^{3/2}}{3}(TA(\theta^2 + aq\mathcal{A})^{-1/2} + q^{-1/2}\mathcal{S}X + q^{-1/2}\mathcal{T})\nabla q \\ T &= 2(\theta^2 + \alpha q^{2/3})^{1/2}q^{-1/2}(\mathcal{S} + \mathcal{T} + 2N\mathcal{A}^{1/2}(1 - \frac{3}{4}B'(\alpha^{3/2}/h))) \end{aligned}$$

therefore, with $\theta = r\omega$ and $|\omega| = 1$, homogeneity in θ yields a parametrization of $\pi(\Lambda_N^{mod})$ by $d + 1$ parameters $(\omega, \mathcal{S}, \mathcal{T})$:

$$\begin{aligned} X &= 1 + \mathcal{T}^2 - \mathcal{S}^2 \\ -y &= a^{1/2}T(1 + aq(1 + \mathcal{T}^2))^{-1/2}\omega + \frac{a^{3/2}}{3}(T(1 + \mathcal{T}^2)(1 + aq(1 + \mathcal{T}^2))^{-1/2})\nabla q \\ &\quad + \frac{a^{3/2}}{3q^{1/2}}(\mathcal{S} + \mathcal{T})(1 + \mathcal{S}\mathcal{T} - \mathcal{S}^2)\nabla q \\ T &= 2(1 + a(1 + \mathcal{T}^2)q)^{1/2}q^{-1/2}(\mathcal{S} + \mathcal{T} + 2N(1 + \mathcal{T}^2)(1 - \frac{3}{4}B'(\mathcal{A}^{3/2}\lambda q^{1/2}))) \end{aligned}$$

where $\lambda = a^{3/2}r/h$.

Remark 3.3. We recover formula (2.11) to (2.14) in [8] when $d = 2$, with small adjustments due to our slightly different choice of ψ_a at present.

The second equation may be rewritten as follows,

$$-y = a^{1/2}(F\omega + aG\nabla q(\omega)),$$

with

$$F = T(1 + aq(\omega)\mathcal{A})^{-1/2}, \quad G = \frac{\mathcal{A}T}{3(1 + aq(\omega)\mathcal{A})^{1/2}} + \frac{\mathcal{S}X + \mathcal{T}}{3q^{1/2}(\omega)}$$

and because parameters $\beta = (\theta, \alpha, s, \varrho)$ are such that $1 - \varepsilon_0 \leq |\theta| \leq 1 + \varepsilon_0$ and $|\alpha|, |s|, |\varrho| \leq \varepsilon_0$, one has $|aG| \leq C\varepsilon_0 F$ and $F \sim T$.

We now return to the general case and perform the same reduction:

$$\begin{aligned}\Psi_N &= \psi_0 + a^{3/2}\Psi_{N,0} + a^{3/2}EA + a^2\Psi_* \\ \psi_0 &= y\Theta + a^{1/2}T|\theta|_{\alpha=0} \\ \Psi_{N,0} &= S^3/3 + S(Xq^{1/3}(\theta|_{\alpha=0})(1+\ell) - A) + \Upsilon^3/3 + \Upsilon q^{1/3}(\theta|_{\alpha=0}) - \Upsilon A \\ &\quad - \frac{4}{3}NA^{3/2} + N\tilde{\lambda}^{-1}B(\tilde{\lambda}A^{3/2}) + TAq^{2/3}(\theta|_{\alpha=0})/(2|\theta|_{\alpha=0}) \\ E &= E_0T + E_1Sx + E_2\Upsilon a\end{aligned}$$

and finally

$$(3.6) \quad \Psi_* = XS^2f + AXf + a^{1/2}X^2Sf + \Upsilon^2f + a^{3/2}A^2\Upsilon f + f + Ta^{1/2}A^2f$$

where $\tilde{\lambda} = a^{3/2}/h$.

Recall homogeneities for $\Psi_N = y\Theta + t|\theta|_{\alpha=0} + \Psi_{N,0} + \varepsilon\alpha + \Psi_*$: (x, y, t) have homogeneity 0, s has $1/3$, α has $2/3$, ϱ has $1/3$ and Θ has 1 as well. The first phase Ψ_0 is the phase related to the free evolution and reads

$$\Psi_0 = y\theta + t\rho(\alpha, \theta) + s^3/3 + s(xq^{1/3}(\theta) - \alpha) + \psi_a(\varrho, \theta) - \varrho\alpha + \rho(\alpha, \theta)\tilde{\Gamma}(x, y, sq^{1/3}/\rho, \theta/\rho)$$

and ψ_a may be chosen homogeneous of degree one in (ϱ, θ) so Ψ_0 has the same homogeneity. We did change variables twice, and both maps are homogeneous: the first one is $G_* : (x, y, t; \Theta, \alpha, s, \varrho) \rightarrow (x, y, t; \theta(y, \alpha, \Theta), \alpha, s, \varrho)$ and the second one is $G^* : (x, y, t; (r, \omega), \tilde{\alpha}, \tilde{s}, \tilde{\varrho}) \rightarrow (x, y, t; r(\omega + \nu_0(y, \omega)), r^{2/3}\tilde{\alpha}, r^{1/3}\tilde{s}, r^{1/3}\tilde{\varrho})$. If one writes $\theta(y, 0, \Theta) = r\omega$ with $|\omega| = 1$, we have $\Theta = r(\omega + \nu_0(y, \omega))$, and $\nu_0 = \int_0^1 \nabla_y B_0(\mu y, \omega) d\mu = O(y^2)$.

Set $\Psi_0^\# = \Psi_0 \circ G^*$, then $\Psi_0^\#$ is Ψ_0 expressed in the variables $x, y, t, r, \omega, \tilde{\alpha}, \tilde{s}, \tilde{\varrho}$. From $\Psi_0 = \psi_0 + \Psi_{0,0} + \varepsilon\alpha + \Psi_*$, we have $\psi_0 = r\tilde{\psi}_0$, $\Psi_{0,0} = r\tilde{\Psi}_{0,0}$, $\varepsilon = r\tilde{\varepsilon}$, with

$$\begin{aligned}\tilde{\psi}_0 &= y(\omega + \nu_0(y, \omega)) + t \\ \tilde{\Psi}_{0,0} &= \tilde{s}^3/3 + \tilde{s}(xq^{1/3}(\omega)(1+\ell)(y, \omega) - \tilde{\alpha}) + \tilde{\varrho}^3/3 + a\tilde{\varrho}q^{1/3}(\omega) \\ &\quad - \tilde{\varrho}\tilde{\alpha} + tq^{2/3}(\omega)\tilde{\alpha}/2 \\ \tilde{\varepsilon} &= \tilde{\varepsilon}_0(y, \omega)t + \tilde{\varepsilon}_1(y, \omega)\tilde{s}x + \tilde{\varepsilon}_2(y, \omega)\tilde{\varrho}a.\end{aligned}$$

As $\tilde{\psi}_0$, $\tilde{\Psi}_{0,0}$ and $\tilde{\varepsilon}$ are independent of r , the same holds true for $\tilde{\Psi}_* = \tilde{\Psi}_0 - \tilde{\psi}_0 - \tilde{\Phi}_{0,0} - \tilde{\varepsilon}\tilde{\alpha}$, where $\tilde{\Psi}_0 = r^{-1}\Psi_0^\#$, and we also have $\tilde{\Psi}_* = r^{-1}\Psi_*$ independent of r .

In a similar way, we have $\Psi_N = \Psi_0 - NhL(\alpha h^{-2/3})$, and writing $\tilde{\Psi}_N = r^{-1}\Psi_N$, $\tilde{\Psi}_{N,0} = \tilde{\Psi}_{0,0} - r^{-1}NhL(\alpha h^{-2/3})$, then $\tilde{\Psi}_N = \tilde{\psi}_0 + \tilde{\Psi}_{N,0} + \tilde{\varepsilon}\tilde{\alpha} + \tilde{\Psi}_*$ with $\tilde{\Psi}_*$ independent of r .

Therefore, after the same change of scale as on the model, we get

$$\begin{aligned}
\Psi_N &= r(\tilde{\psi}_0 + a^{3/2}\tilde{\Psi}_{N,0} + a^{3/2}\tilde{E}A + a^2\tilde{\Psi}_*) \\
\tilde{\psi}_0 &= y(\omega + \nu_0(y, \omega)) + t \\
\tilde{\Psi}_{N,0} &= S^3/3 + S(Xq^{1/3}(\omega)(1 + \ell)(y, \omega) - A) + \Upsilon^3/3 + \Upsilon(q^{1/3}(\omega) - A) \\
&\quad - \frac{4}{3}NA^{3/2} + N\lambda^{-1}B(\lambda A^{3/2}) + TAq^{2/3}(\omega)/2 \\
\tilde{E} &= \tilde{E}_0T + \tilde{E}_1Sx + \tilde{E}_2\Upsilon a,
\end{aligned}$$

where for $k = 0, 1, 2$, $\tilde{E}_k = O(y)$ as functions of (y, ω) , and $\lambda = a^{3/2}r/h$. Moreover, from (3.6), we see that Ψ_* is independent of r . The phase $\Psi_{N,0}$ depends on r only through the parameter λ . Now,

as $\frac{\partial \theta}{\partial \Theta} = I + O(y^2)$, we see that Λ_N can be parametrized not only by $\Theta, \alpha, s, \varrho$ but also by $\theta, \alpha, s, \varrho$. Let us define

$$Q = a^{-1/2} \frac{\partial}{\partial \theta} (\Psi_N - \Psi_N^{mod}),$$

and assume from now on that $T \geq T_0 > 0$. Denote by $\tilde{K}_0 = \{(\alpha, s, \varrho, \tilde{X}, \tilde{a}, r) | \alpha = s = \varrho = \tilde{a} = 0\}$ and by \tilde{O} the set of smooth functions near $\{x = y = t = 0\} \times \tilde{K}_0$ which moreover vanish on \tilde{K}_0 .

We will prove that $T^{-1}Q \in \tilde{O}$, provided that $\tilde{X} = a^{1/2}X$, $\tilde{T} = T^{-1}$ and $\tilde{a} = a^{1/2}$.

We have

$$Q = aSx\partial_\theta(q^{1/3}\ell) + aA(T\partial_\theta E_0 + Sx\partial_\theta E_1 + \Upsilon a\partial_\theta E_2) + a^{3/2}\partial_\theta \Psi_*.$$

Then, one has

$$aS\partial_\theta(q^{1/2}\ell) = (a^{1/2}S)(a^{1/2}X)\partial_\theta(q^{1/3}\ell) = s\tilde{X}\partial_\theta(q^{1/3}\ell) \in \tilde{O},$$

as well as

$$\begin{aligned}
aTA\partial_\theta \tilde{E}_0 &= \alpha T\partial_\theta \tilde{E}_0 \in T\tilde{O} \\
aASx\partial_\theta \tilde{E}_1 &= \alpha s\tilde{X}\partial_\theta \tilde{E}_1 \in \tilde{O} \\
aA\Upsilon a\partial_\theta \tilde{E}_2 &= \alpha \tilde{a}\varrho\partial_\theta \tilde{E}_2 \in \tilde{O}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
a^{3/2}XS^2f &= (a^{1/2}X)(a^{1/2}S)^2f = \tilde{X}s^2f \in \tilde{O}, \\
a^{3/2}AXf &= \alpha\tilde{X}f \in \tilde{O}, \\
a^{3/2}X^2f &= \tilde{X}^2\tilde{a}f \in \tilde{O}, \\
a^{3/2}f &= \tilde{a}^3f \in \tilde{O}, \\
a^{3/2}\Upsilon^2f &= \tilde{a}\varrho^2f \in \tilde{O} \\
a^3A^2\Upsilon f &= \alpha^2\varrho\tilde{a}f \in \tilde{O}, \\
a^{3/2}TA^2a^{1/2}f &= T\alpha^2f \in T\tilde{O}.
\end{aligned}$$

But obviously, $\tilde{O} \subset T\tilde{O}$ and therefore we proved $Q \in T\tilde{O}$. \square

Recalling that

$$\begin{aligned}
H^{mod} &= \frac{1}{3}\mathcal{A}a + \frac{a}{3} \frac{\mathcal{S}X + \mathcal{T}(1 + aq\mathcal{A})^{1/2}}{q^{1/2}(\omega)} \frac{1}{T} \\
F &= T(1 + aq\mathcal{A})^{1/2}
\end{aligned}$$

we proved earlier that on $\pi(\Lambda_N^{mod})$, we have $-y = a^{1/2}F(\omega + H^{mod})$. We also have $F \sim T$ and $H^{mod} \in \tilde{O}$.

In the general case, indeed we have $-y = a^{1/2}F(\omega + H^{mod} + Q/F)$, and since $Q/F \in \tilde{O}$, setting $G_1 = H^{mod} + Q/F$ we have on $\pi(\Lambda_N)$ that $-y = a^{1/2}F(\omega + G_1)$ where $G_1 \in \tilde{O}$. Therefore,

$$\Omega = \frac{y}{|y|} = -\omega + H_1 \quad \text{with } H_1 = G_1(1-K) - \omega K, \quad \text{where } K = 1 - \frac{1}{(1 + K_1)^{1/2}}, \quad K_1 = |G_1|^2 + 2\omega \cdot G_1.$$

We can then remark that $|H_1| + |\partial_\omega H_1|$ is small near $K_0 = \{x = y = t = 0\} \times \tilde{K}_0$ and apply the inverse function theorem to the map $(w, \omega) \rightarrow (w, \Omega(w, \omega) = -\omega + H_1(w, \omega))$ near K_0 . The inverse map is $(w, \Omega) \rightarrow (w, \omega(w, \Omega))$.

Lemma 3.4. *Let $(x, y, t) \in \pi(\Lambda_N), (x', y', t') \in \pi(\Lambda_{N'})$, $|N| \geq 1, |N'| \geq 1$, r_0 small and*

$$\begin{aligned}
(x', y', t') \in B_0(x, y, t) &= \{(x', y', t') | 0 \leq x' \leq a, |y' - y| \leq r_0 a^{1/2}, \\
&\quad \left| \frac{y}{|y|} - \frac{y'}{|y'|} \right| \leq r_0 a^{1/2}, |t - t'| \leq r_0 a^{1/2}\}
\end{aligned}$$

and let $\beta = (\alpha, s, \tilde{t}, \theta)$ and $\beta' = (\alpha', s', \tilde{t}', \theta')$ be parameter values on Λ_N and $\Lambda_{N'}$. Then

$$TA^{-1/2}|\omega - \omega'| = O(1).$$

Recall that $\omega = \omega(w, \Omega)$ and $\omega' = \omega(w', \Omega')$ and we proceed by using Taylor's formula. From our hypothesis $|\Omega - \Omega'| \leq r_0 a^{1/2}$, we obviously have $TA^{-1/2}|\Omega - \Omega'| = O(1)$. As already said, $w = (\alpha, s, \tilde{t}, a^{1/2}X, ry, 1/T)$. We check first $TA^{-1/2}|1/T - 1/T'| = O(1)$, as $|T - T'| \leq r_0$. Moreover, $TA^{-1/2}a^{1/2}|X - X'| = O(1)$, and $TA^{-1/2}|s - s'| = O(1)$ as, on

both Lagrangians Λ_N and $\Lambda_{N'}$, $|S|A^{-1/2}$ and $|S'|(A')^{-1/2}$ are bounded and $A^{1/2} \sim A'^{1/2}$. Actually, one has $T(A^{-1/2} - A'^{-1/2}) = O(1)$ by the proof of Lemma 3.18 in [8]. Similarly, one gets $TA^{-1/2}|\tilde{t} - \tilde{t}'| = O(1)$ and then $TA^{-1/2}|\alpha - \alpha'| = TaA^{-1/2}|A - A'| = O(1)$.

It remains to deal with $TA^{-1/2} \sup |\partial_r \omega|$. Write $\partial_r \omega = (\partial_\omega \Omega)^{-1} \partial_r \Omega$, we compute

$$\partial_r \Omega = -(H^{mod} + Q/F) \partial_r K + (1 - K) \partial_r (Q/F) - \omega \partial_r K.$$

as $\partial_r H^{mod} = 0$. We also have $TA^{-1/2} \partial_r (Q/F) = O(1)$ so it remains to check that $TA^{-1/2} \partial_r K = O(1)$. But

$$TA^{-1/2} \partial_r K = -\frac{1}{2} TA^{-1/2} \partial_r K_1 (1 + K_1)^{-3/2} = \frac{1}{2} TA^{-1/2} \partial_r (Q/F) (G_1 + \omega) (1 + K_1)^{-3/2} = O(1).$$

This concludes the proof. \square

Definition 3.5. Let us define

$$\begin{aligned} \mathcal{N}(x, y, t) &= \{N \in \mathbb{Z} \mid |N| \geq 1, (x, y, t) \in \pi(\Lambda_N)\} \\ \mathcal{N}_1(x, y, t) &= \{N \in \mathbb{Z} \mid |N| \geq 1, B_0(x, y, t) \cap \pi(\Lambda_N) \neq \emptyset\}, \end{aligned}$$

where $B_0(x, y, t)$ is a suitable neighborhood of (x, y, t) which was introduced earlier.

Let $K_0 > 0$ be a large constant to be specified later. We will consider two cases,

$$(3.7) \quad \forall N \in \mathcal{N}_1(x, y, t), \text{ we have } T/|N| \leq K_0$$

or

$$(3.8) \quad \exists N \in \mathcal{N}_1(x, y, t), \text{ such that } T/|N| \geq K_0.$$

One may then reproduce the proofs of Lemma 2.18 and 2.19 in [8] (using the previous lemma) and obtain the alternative

- if (3.7) holds, then

$$|\mathcal{N}_1(x, y, t)| \leq \phi(x, y, t)$$

where $\phi(x, y, t) = C + CT\lambda^{-2}$;

- if (3.8) holds, then

$$(3.9) \quad |\mathcal{N}_1(x, y, t)| \leq \Phi(x, y, t)$$

where $\Phi(x, y, t) = C + CT\lambda^{-2}(T/|N|)^{-2(3+1/2)}$;

Notice that the second case is an improvement of the first one, as $T/|N| \geq K_0$ is large.

Lemma 3.6. *Outside of the set $\mathcal{N}_1(x, y, t)$ the parametrix is smooth, e.g.*

$$\sum_{N \notin \mathcal{N}_1(x, y, t)} V_N(x, y, t) \in O_{C^\infty}(h^\infty).$$

The lemma may be exactly proved as in [8]. \square

We now introduce a splitting of V_N for $|N| \geq 2$: $V_N = V_{N,1} + V_{N,2}$ where on the support of the integrand of $V_{N,1}$ we have $A \geq A_1/2$ and on the support of the integrand of $V_{N,2}$ we have $A \leq 2A_1$, where $A_1 > 0$ is a large constant. Before stating estimates for V_N , when $|N| \geq 1$, we recall that we have the free space dispersion for V_0 , assuming that $t > h$ so that the dispersive effect takes over:

$$|V_0(x, y, t)| \leq Ch^{-d} \left(\frac{h}{t} \right)^{\frac{d-1}{2}}.$$

Proposition 3.7. *Assume $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, and $h \in (0, h_0]$. Then there exists $C(\epsilon, a_0, h_0)$ such that, with $0 \leq x \leq 2a$,*

$$|V_1(x, y, t)| \leq Ch^{-d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \left(\left(\frac{h}{t} \right)^{1/2} + a^{1/4} \left(\frac{h}{t} \right)^{1/4} + h^{1/3} \right).$$

Proposition 3.8. *Assume $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, and $h \in (0, h_0]$. Then there exists $C(\epsilon, a_0, h_0)$ such that*

$$\left| \sum_{|N| \geq 2} V_{N,1}(x, y, t) \right| \leq Ch^{-d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} h^{1/3}.$$

Proposition 3.9. *Assume $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, and $h \in (0, h_0]$. Then there exists $C(\epsilon, a_0, h_0)$ such that, with $0 \leq x \leq 2a$,*

$$\left| \sum_{|N| \geq 2} V_{N,2}(x, y, t) \right| \leq Ch^{-d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \left(a^{1/4} \left(\frac{h}{t} \right)^{1/4} + h^{1/3} \right).$$

Our main estimate, (1.3) from Theorem 1.1, follows at once from the previous propositions, in the regime $a \geq h^{2/3-\epsilon}$, together with the symmetry properties of the Green function (exchanging source and observation points in space-time).

3.1. The main term $V_{N,2}$. Recall that $V_{N,2}$ is an oscillatory integral associated to the phase

$$\Psi_N = r(\tilde{\psi}_0 + a^{3/2}\tilde{\Psi}_{N,0} + a^{3/2}\tilde{E}\mathcal{A} + a^2\tilde{\Psi}_*)$$

where, with $\lambda = a^{3/2}r/h$,

$$\begin{aligned}\tilde{\psi}_0 &= (\omega + \nu_0(y, \omega))y + t \\ \tilde{\Psi}_{N,0} &= \mathcal{S}^3/3 + \mathcal{S}(Xq^{1/3}(\omega)(1 + \ell)(y, \omega) - \mathcal{A}) + \mathcal{T}^3/3 + \mathcal{T}q^{1/3}(\omega) \\ &\quad - \mathcal{T}\mathcal{A} + T\mathcal{A}q^{2/3}(\omega)/2 - \frac{4}{3}N\mathcal{A}^{3/2} + NB(\lambda\mathcal{A}^{3/2})/\lambda, \\ \tilde{\Psi}_* &= X\mathcal{S}^2f + \mathcal{A}Xf + a^{1/2}X^2\mathcal{S}f + \mathcal{T}^2f + a^{3/2}\mathcal{A}^2\mathcal{T}f + f + Ta^{1/2}\mathcal{A}^2f \\ \tilde{E} &= \tilde{E}_0T + \tilde{E}_1\mathcal{S}x + \tilde{E}_2\mathcal{T}a,\end{aligned}$$

where the $\tilde{E}_{0,1,2}$ are $O(y)$ functions of (y, ω) .

Recall as well that

$$V_{N,2} = \frac{a^2}{h^{d+1}} \int q_{h,2} e^{\frac{i}{h}\Psi_N} d\Theta d\mathcal{S} d\Upsilon dA$$

where $q_{h,2}$ is a symbol of degree zero, and up to an harmless modification of $q_{h,2}$ we may replace $d\mathcal{S}d\Upsilon dA$ by $d\mathcal{S}d\mathcal{T}d\mathcal{A}$. Given that on the support of $V_{N,2}$ we have $|\mathcal{A}| \leq A_2$, where A_2 is a (large) constant depending on constant A_1 from the definition of $V_{N,2}$, we have, with y close to zero and $\sup(|\mathcal{S}|, |\mathcal{T}|) \geq M_0$, M_0 large,

$$\begin{aligned}|\nabla_{\mathcal{S},\mathcal{T}}\tilde{\Psi}_*| &\leq C(1 + |\mathcal{S}| + |\mathcal{T}|) \\ |\mathcal{A}||\nabla_{\mathcal{S},\mathcal{T}}\tilde{E}| &\leq C\varepsilon_0 \\ |\nabla_{\mathcal{S},\mathcal{T}}\Psi_N| &\geq c_0a^{3/2}(1 + |\mathcal{S}| + |\mathcal{T}|)^2.\end{aligned}$$

Integrating by parts, we may therefore further restrict the support of $V_{N,2}$ to a compact region on $(\mathcal{S}, \mathcal{T})$, up to a remainder in $O_{C^\infty}(h^\infty)$.

Moreover, the compact support in $\mathcal{A}, \mathcal{S}, \mathcal{T}$, $0 \leq X \leq 1$ readily imply that $a^2\tilde{\Psi}_*$ is a perturbation of the model phase.

We now wish to perform a stationnary phase with respect to ω . The phase ψ_0 has two non degenerate critical points. At these points, one has

$$\nabla_\omega^2\psi_0 = |y|O(1), \quad (\nabla_\omega^2\psi_0)^{-1} = |y|^{-1}O(1).$$

Moreover, in view of Lemma 3.2 and its proof, one may assume that $|y| \geq c_0t$, $|t| \geq c_1a$, and in the computation of $a^{3/2}\nabla_\omega^2\tilde{\Psi}_{N,0}$, the worst term is controled as follows

$$|(a^{1/2}T)(a\mathcal{A})\nabla_\omega^2(q^{2/3})| \leq C\varepsilon_0|y|,$$

as $t = a^{1/2}T \leq C_0|y|$ and $a|\mathcal{A}| \leq \varepsilon_0$ on the support of the integrand. For other terms,

$$|a^{3/2}\mathcal{S}X\nabla_\omega^2((1 + \ell)q^{1/3})| + |a^{3/2}\mathcal{T}\nabla_\omega^2(q^{1/3})| \leq C\varepsilon_0a \leq C\varepsilon_0|y|,$$

and therefore we get

$$|a^{3/2}\nabla_\omega^2\tilde{\Psi}_{N,0}| \leq C\varepsilon_0|y|.$$

We also have

$$|a^{3/2}\nabla_\omega^2(E\mathcal{A})| \leq C|y|a|\mathcal{A}| \leq C\varepsilon_0|y|,$$

and, given $a^{3/2}\nabla_\omega^2\tilde{\psi}_1 \in T\tilde{O}$ and $a^{1/2}T = t \leq C_0^{-1}|y|$,

$$|a^2\nabla_\omega^2\tilde{\psi}_1| \leq C\varepsilon_0t \leq C\varepsilon_0|y|.$$

We have obtained that

$$\nabla_\omega^2\tilde{\Psi}_N = \nabla_\omega^2\tilde{\psi}_0 + O(1)\varepsilon_0|y|,$$

and we may indeed apply stationary phase in $\omega \in \mathbb{S}^{d-2}$, which yields a decay factor

$$(3.10) \quad \left(\frac{h}{r|y|}\right)^{\frac{d-2}{2}} \leq C \left(\frac{h}{t}\right)^{\frac{d-2}{2}}$$

as $|y| \geq c_0t$ and $r \sim 1$.

Now, the next step is to apply stationary phase with respect to \mathcal{A} , at least when $N \neq 0$. Let us define

$$\Psi_{N,c} = \Psi_N|_{\omega=\omega_c}$$

where ω_c is a critical point of the previous step. One has

$$\begin{aligned} \partial_{\mathcal{A}}\Psi_{N,c} &= \partial_{\mathcal{A}}\Psi_N|_{\omega=\omega_c} \\ \partial_{\mathcal{A}}^2\Psi_{N,c} &= \partial_{\mathcal{A}}^2\Psi_N|_{\omega=\omega_c} + \partial_{\mathcal{A},\omega}^2\Psi_N|_{\omega=\omega_c}\partial_{\mathcal{A}}\omega_c \end{aligned}$$

we also have $\partial_{\mathcal{A}}\omega_c = -(\nabla_\omega^2\Psi_N)^{-1}\partial_{\mathcal{A},\omega}^2\Psi_N$ and $\nabla_\omega^2\Psi_N = M$ with $M = |y|O(1)$ and $M^{-1} = |y|^{-1}O(1)$, for small $y \neq 0$, $|y| \geq c_0t$ and $t \geq c'_0a$.

Moreover,

$$\partial_{\mathcal{A},\omega}^2\Psi_N = a^{3/2}T\partial_\omega(q^{2/3}(\omega)/2) + a^{3/2}\partial_\omega E + a^2\partial_{\omega,\mathcal{A}}^2\psi_1,$$

and as $\partial_{\mathcal{A}}\psi_1 = f + sXf + tAf$, we get

$$|\partial_{\mathcal{A},\omega}^2\Psi_N| \leq Cta \leq Ca|y|$$

on the domain of integration. Therefore $|\partial_{\mathcal{A}}\omega_c| \leq Ca$, and from

$$(3.11) \quad |\partial_{\mathcal{A}}^2\Psi_N| \geq c_0a^{3/2}|N|\mathcal{A}^{-1/2}$$

we get

$$(3.12) \quad |\partial_{\mathcal{A}}^2\Psi_{N,c}| \geq c_0a^{3/2}|N|\mathcal{A}^{-1/2}(1 - Ca^{1/2}\mathcal{A}^{1/2}/|N|)$$

where we assumed $N \neq 0$ and $a\mathcal{A}$ is small, so that the second term on the righthand side is irrelevant.

Hence we may apply stationnary phase in \mathcal{A} and get another decay factor

$$(3.13) \quad \frac{1}{\sqrt{a^{3/2}|N|}} \times \frac{1}{\sqrt{1/h}} \lesssim \frac{1}{(\tilde{\lambda}|N|)^{1/2}}.$$

Let $F_N = \Psi_{N|(\omega, A)=(\omega_c, A_c)}$ be the phase at the critical points and recall that we assume $N \neq 0$ and $1 \leq |N| \leq c_0 a^{-1/2}$. At the critical points we have

$$\begin{aligned} \partial_S F_N &= a^{3/2} r \partial_S \Psi_{N,0}|_{A_c, \omega_c} + a^2 r \partial_S \tilde{\psi}_1|_{A_c, \omega_c} + a^{3/2} r \mathcal{A}_c \partial_S \tilde{E}|_{\omega_c} \\ \partial_T F_N &= a^{3/2} r \partial_T \Psi_{N,0}|_{A_c, \omega_c} + a^2 r \partial_T \tilde{\psi}_1|_{A_c, \omega_c} + a^{3/2} r \mathcal{A}_c \partial_T \tilde{E}|_{\omega_c}. \end{aligned}$$

Recall that

$$\partial_S \Psi_{N,0} = \mathcal{S}^2 + Xq^{1/3}(\omega)(1 + \ell)(y, \omega) - \mathcal{A}, \quad \partial_T \tilde{\Psi}_{N,0} = \mathcal{T}^2 + Xq^{1/3}(\omega) - \mathcal{A}.$$

We now compute \mathcal{A}_c more explicitly: \mathcal{A}_c is a solution to

$$\partial_{\mathcal{A}} \tilde{\Psi}_{N,0} + a^{1/2} \partial_{\mathcal{A}} \tilde{\psi}_1 + \tilde{E} = 0$$

and

$$\begin{aligned} \partial_{\mathcal{A}} \tilde{\Psi}_{N,0} &= -(\mathcal{S} + \mathcal{T}) - 2N\mathcal{A}^{1/2} + \frac{3}{2}NB'(\lambda\mathcal{A}^{3/2})\mathcal{A}^{1/2} + \frac{1}{2}q^{2/3}(\omega)T \\ \partial_{\mathcal{A}} \psi_1 &= f + sXf + t\mathcal{A}f, \quad \tilde{E} = \tilde{E}_0T + \tilde{E}_1\mathcal{S}x + \tilde{E}_2a\mathcal{T}. \end{aligned}$$

Then \mathcal{A}_c is a solution to the equation (in \mathcal{A})

$$\begin{aligned} \mathcal{A}^{1/2}(1 - \frac{3}{4}B'(\lambda\mathcal{A}^{3/2})) &= q^{2/3}(\omega)\frac{T}{4N} - \frac{1}{2}(\frac{\mathcal{S}}{N} + \frac{\mathcal{T}}{N}) \\ &\quad + \frac{1}{2}(\frac{T}{N}\tilde{E}_0 + \frac{\mathcal{S}}{N}x\tilde{E}_1 + a\frac{\mathcal{T}}{N}\tilde{E}_2) + \frac{a^{1/2}}{2N}(f + sXf + t\mathcal{A}f) \end{aligned}$$

which we rewrite as

$$\begin{aligned} \mathcal{A}^{1/2}(1 - \frac{3}{4}B'(\lambda\mathcal{A}^{3/2})) &= (q^{2/3}(\omega_c) + 2\tilde{E}_0(y, \omega_c))\frac{T}{4N} - \frac{\mathcal{S}}{2N}(1 - 2x\tilde{E}_1 - xf) \\ &\quad - \frac{\mathcal{T}}{2N}(1 - 2a\tilde{E}_2) + \frac{a^{1/2}}{2N}(f + t\mathcal{A}f) \end{aligned}$$

and from $t = a^{1/2}T$, we get, with $\alpha_c = a\mathcal{A}_c$,

$$\begin{aligned} \mathcal{A}_c^{1/2}(1 - \frac{3}{4}B'(\lambda\mathcal{A}_c^{3/2})) &= (q^{2/3}(\omega_c) + 2\tilde{E}_0(y, \omega_c) + 2\alpha_cf)\frac{T}{4N} - \frac{\mathcal{S}}{2N}(1 - 2x\tilde{E}_1 - xf) \\ &\quad - \frac{\mathcal{T}}{2N}(1 - 2a\tilde{E}_2) + \frac{a^{1/2}}{2N}f \end{aligned}$$

where ω_c depends on $(\mathcal{A}, \mathcal{S}, \mathcal{T})$ and α_c depends on $(\mathcal{S}, \mathcal{T})$. We also have (3.11) at $\mathcal{A} = \mathcal{A}_c$ from (3.12), modifying the constant c_0 , and we have at most one critical point \mathcal{A}_c on the support in \mathcal{A} of $q_{h,2}$, which is a fixed compact set $K_0 = [c, C] \subset \mathbb{R}_+$.

Therefore \mathcal{A}_c is a solution to

$$\mathcal{A}_c^{1/2}(1 - \frac{3}{4}B'(\lambda\mathcal{A}_c^{3/2})) = \tilde{\alpha}_0 \frac{T}{4N} - (\tilde{\beta}_0 \frac{\mathcal{S}}{2N} + \tilde{\gamma}_0 \frac{\mathcal{T}}{2N}) + \frac{a^{1/2}}{2N}f$$

where $\tilde{\alpha}_0 = (q^{2/3}(\omega_c) + 2\tilde{E}_0(y, \omega_c) + 2\alpha_c f)$, $\tilde{\beta}_0 = (1 - 2x\tilde{E}_1 - xf)(\alpha_c, \omega_c)$ and $\tilde{\gamma}_0 = (1 - 2a\tilde{E}_2)(y, \omega_c)$.

Recall we are on a region where $|y| > c_0 t$, $t > c_1 \sqrt{a}$, $|(\mathcal{S}, \mathcal{T})| < M_0$ and therefore, for small $|y|$ and a , we have $\tilde{\alpha}_0 \geq c_0 > 0$ and $\mathcal{A}_c \in K_1$, where K_1 is a neighborhood of K_0 , implies that $T/(4|N|) \leq C_0$. Therefore, in such a case, we have

$$(3.14) \quad \frac{1}{C_0} \leq \frac{T}{4|N|} \leq C_0.$$

On the other hand, if $\text{dist}(\mathcal{A}_c, K_0) \geq \delta > 0$, we get that $V_{N,2} \in O(h^\infty |N|^{-\infty})$: we have

$$|\partial_{\mathcal{A}} \Psi_N| = |\partial_{\mathcal{A}} \Psi_N - \partial_{\mathcal{A}} \Psi_{N,|\mathcal{A}=\mathcal{A}_c}| = \left| \int_0^1 \partial_{\mathcal{A}^2} \Psi_N(\mathcal{A}_c + s(\mathcal{A} - \mathcal{A}_c)) ds \right| |\mathcal{A} - \mathcal{A}_c|$$

and therefore, for $\mathcal{A} \in K_0$,

$$|\partial_{\mathcal{A}} \Psi_N| \geq c_0 a^{3/2} |N| \delta$$

and each integration by part with vector field $L = {}^t L = \frac{h}{\partial_{\mathcal{A}} \Psi_N(\omega_c)} \partial_{\mathcal{A}}$ yields a factor $ha^{-1/2} |N|^{-1} \leq h^{1/3} |N|^{-1}$ as $a \geq h^{3/2}$.

We now rewrite the equation for \mathcal{A}_c by Taylor expanding B' , so that

$$\mathcal{A}_c^{1/2} = \tilde{\alpha}_0 \frac{T}{4N} - (\tilde{\beta}_0 \frac{\mathcal{S}}{2N} + \tilde{\gamma}_0 \frac{\mathcal{T}}{2N}) + \frac{a^{1/2}}{2N}f + \frac{g_0}{\lambda^2}$$

where g_0 is a zeroth order symbol in λ , $\tilde{\alpha}_0$, $\tilde{\beta}_0$ and $\tilde{\gamma}_0$ depend on \mathcal{S}, \mathcal{T} through (ω_c, α_c) and we set $\tilde{f} = f|_{\omega_c, \alpha_c}$. If we Taylor expand \tilde{f} on $s = \varrho = 0$, we get

$$\tilde{f} = \sum_{j < M} a^{j/2} N^j \tilde{f}_j + N^M a^{M/2} O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^M),$$

and

$$\tilde{f}_j = \sum_{k+l=j} f_{k,l}^j (\frac{\mathcal{S}}{N})^k (\frac{\mathcal{T}}{N})^l,$$

where $f_{k,l}^j$ is independent of \mathcal{S}, \mathcal{T} and $j = 0, 1, \dots, M-1$.

Moreover, since we have $\partial_{\mathcal{S}} \omega_c, \partial_{\mathcal{T}} \omega_c = O(a)$, $\nabla_{\mathcal{S}, \mathcal{T}}^2 \omega_c = O(a^{3/2})$ over our domain, we have also $\partial_{\mathcal{S}} \alpha_c, \partial_{\mathcal{T}} \alpha_c = O(a/N)$, $\nabla_{\mathcal{S}, \mathcal{T}}^2 \alpha_c = O(a^{3/2}/N)$, so that

$$\alpha_c = \alpha_0 + a(\alpha_1 \frac{\mathcal{S}}{N} + \alpha_2 \frac{\mathcal{T}}{N}) + Na^{3/2} O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2)$$

and

$$\omega_c = \omega_0 + aN(\bar{\omega}_1 \frac{\mathcal{S}}{N} + \bar{\omega}_2 \frac{\mathcal{T}}{N}) + N^2 a^{3/2} O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2)$$

as well as

$$\tilde{f} = \tilde{f}_0 + aN(\tilde{f}_1 \frac{\mathcal{S}}{N} + \tilde{g}_1 \frac{\mathcal{T}}{N}) + N^2 a^{3/2} O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2)$$

and $\tilde{f}_0, \tilde{f}_1, \tilde{g}_1$ smooth functions that do not depend on \mathcal{S}, \mathcal{T} .

We now rewrite the equation on \mathcal{A}_c as

$$\mathcal{A}_c^{1/2} = \tilde{a}_0 \frac{T}{4N} - (\tilde{b}_0 \frac{\mathcal{S}}{2N} + \tilde{c}_0 \frac{\mathcal{T}}{2N}) + O_2 + \frac{g_0}{\lambda^2},$$

and $\tilde{a}_0, \tilde{b}_0, \tilde{c}_0$ smooth functions that do not depend on \mathcal{S}, \mathcal{T} , as well as $O_2 = (Na)O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2)$: indeed, we have

$$\mathcal{A}_c^{1/2} = \mathbb{F}_0 + \mathbb{F}_1 + O_2 + g_0 \lambda^{-2},$$

with

$$\mathbb{F}_1 = -(\beta_1 \frac{\mathcal{S}}{2N} + \gamma_1 \frac{\mathcal{T}}{2N})$$

by definition \mathbb{F}_0 is independent of $(\mathcal{S}/N, \mathcal{T}/N)$ while \mathbb{F}_1 is linear in $(\mathcal{S}/N, \mathcal{T}/N)$, and

$$O_2 = (N^2 a^{3/2} + xNA + a^2 N) O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2) \in (Na) O((\frac{\mathcal{S}}{N}, \frac{\mathcal{T}}{N})^2)$$

as $0 \leq x \leq a$, $0 < a < a_0 < 1$ and $|N|a^{1/2} \leq c_0$.

Finally, if we square the above expression,

$$(3.15) \quad \mathcal{A}_c = \mathbb{G}_0 + \mathbb{G}_1 + \tilde{O}_2 + g_1 \lambda^{-2},$$

where $\mathbb{G}_0 = \mathbb{F}_0^2$ is independent of $(\mathcal{S}/N, \mathcal{T}/N)$ while and $\mathbb{G}_1 = 2\mathbb{F}_0\mathbb{F}_1$ is linear in $(\mathcal{S}/N, \mathcal{T}/N)$, $\tilde{O}_2 = \mathbb{F}_1^2 + O_2(2\mathbb{F}_0 + 2\mathbb{F}_1 + O_2)$. Hence, $\tilde{O}_2 = \mathbb{F}_1 + a^{1/2} O((\mathcal{S}/N, \mathcal{T}/N)^2)$, since $\mathbb{F}_0 O_2 = (Na) O((\mathcal{S}/N, \mathcal{T}/N)^2)$, $\mathbb{F}_1 O_2 = a^{1/2} O((\mathcal{S}/N, \mathcal{T}/N)^2)$, and $O_2^2 = a O((\mathcal{S}/N, \mathcal{T}/N)^2)$.

Let $\underline{\chi}$ be a classical symbol of degree zero in λ for $\lambda \geq \lambda_0$ which has moreover compact support in $(\mathcal{S}, \mathcal{T})$, we are left with estimating the following integral

$$\int e^{\frac{i}{h} F_N} \underline{\chi} d\mathcal{S} d\mathcal{T}$$

and, as in [8] we will have to separate between $|N| > \lambda^{1/3}$ and $|N| \leq \lambda^{1/3}$. We start with the first case, for which we have to perform an additional stationary phase with respect to r , in the case where $|N| \geq \lambda$.

Indeed, we abuse notation and set (again) $\Phi_{N,c} = \Psi_N|_{\omega=\omega_c, \mathcal{A}=\mathcal{A}_c}$ and we need to compute its derivatives with respect to r . Observe that ω_c is independent of r as $\Psi_N = r\tilde{\Psi}_N$ and $\partial_\omega \Psi_N = 0 \Leftrightarrow \partial_\omega \tilde{\Psi}_N = 0 \Leftrightarrow \partial_\omega \tilde{\Psi}_0 = 0$.

Now, set $\Psi_N = r\Psi_N^o$ and $\Phi_N = r\Phi_N^o$. As \mathcal{A}_c is a critical point for Φ_N^o , we have

$$\begin{aligned}\partial_r^2 \Phi_N &= \frac{1}{r} \partial_r (r^2 \partial_r \Phi_N^o), \quad \partial_r \Phi_N^o = \partial_r \Psi_N^o|_{\mathcal{A}=\mathcal{A}_c}, \\ \partial_r \Psi_N^o &= a^{3/2} \partial_r \tilde{\Psi}_{N,0} + a^{3/2} \partial_r (\tilde{E}\mathcal{A}) + a^2 \partial_r^2 \tilde{\Psi}_*, \\ r^2 \partial_r \Phi_N^o &= a^{3/2} r^2 \partial_r \tilde{\Psi}_{N,0} + a^2 r^2 \partial_r \tilde{\Psi}_*,\end{aligned}$$

and as $\partial_r(r^2 F) = \partial_\lambda(\lambda^2 F)$ as well as $\partial_\lambda \tilde{\Psi}_{N,0} = N\lambda^{-2} B_1(\lambda \mathcal{A}^{3/2})$ with $B_1(V) = -B(V) + VB'(V)$, we obtain

$$\partial_r^2 \Phi_N = \frac{N}{r} a^{3/2} U_c B''(U_c) \partial_\lambda U_c, \quad \text{with } U_c = \lambda \mathcal{A}_c^{3/2}.$$

From $|\partial_\lambda \mathcal{A}_c^{3/2}| \leq C \mathcal{A}^{-3/2} \lambda^{-3}$ we obtain

$$|\partial_\lambda U_c| = |\lambda \partial_\lambda \mathcal{A}_c^{3/2} + \mathcal{A}_c^{3/2}| \geq c_0 \mathcal{A}_c^{3/2} (1 - C \mathcal{A}^{-3/2} \lambda^{-3}) \geq c_0 \mathcal{A}_c^{3/2} / 2.$$

On the other hand, we have $|B''(U)| \geq c_0 U^{-3}$ for $U \geq U_0 > 0$, as $b_1 = 5/24 \neq 0$ (asymptotic expansion). Finally, we get

$$(3.16) \quad |\partial_r^2 \Phi_N| \geq c_0 |N| \lambda^{-2} \mathcal{A}_c^{-3/2} a^{3/2},$$

and stationary phase in r will produce a factor $\Upsilon^{-1/2}$ with

$$(3.17) \quad \Upsilon = \frac{|N|}{\lambda^2} \frac{a^{3/2}}{h} \mathcal{A}_c^{-3/2},$$

and $\Upsilon \sim |N| \lambda^{-1} \mathcal{A}_c^{-3/2}$, which is of course of interest only in the regime where $|N| \geq c\lambda$.

Lemma 3.10. *If $|N| \geq \lambda^{1/3}$, there exists C (independent of N) such that*

$$\left| \int e^{\frac{i}{h} F_N} \underline{\chi} d\mathcal{S} d\mathcal{T} \right| \leq \frac{C}{\lambda^{2/3}}.$$

We start by rescaling variables, $\mathcal{S} = \lambda^{-1/3} \mathbf{y}$ and $\mathcal{T} = \lambda^{-1/3} \mathbf{x}$, so that we are left with proving

$$(3.18) \quad \left| \int e^{\frac{i}{h} F_N(\lambda^{-1/3} \mathbf{x}, \lambda^{-1/3} \mathbf{y})} \underline{\chi}(\lambda^{-1/3} \mathbf{x}, \lambda^{-1/3} \mathbf{y}) d\mathbf{x} d\mathbf{y} \right| \leq C.$$

From the compact support of $\underline{\chi}$ (and λ large), we obviously have, for any multi-index μ ,

$$|\partial_{(\mathbf{x}, \mathbf{y})}^\mu \underline{\chi}(\lambda^{-1/3} \mathbf{x}, \lambda^{-1/3} \mathbf{y})| \leq C_\mu (1 + |\mathbf{x}| + |\mathbf{y}|)^{-|\mu|}.$$

Compute

$$\begin{aligned}\partial_{\mathcal{S}}F_N &= a^{3/2}r_c(\mathcal{S}^2 + Xq^{1/3}(1 + \ell) - \mathcal{A}_c(1 - x\tilde{E}_1) + a^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*) \\ \partial_{\mathcal{T}}F_N &= a^{3/2}r_c(\mathcal{T}^2 + q^{1/3} - \mathcal{A}_c(1 - a\tilde{E}_2) + a^{1/2}\partial_{\mathcal{T}}\tilde{\Psi}_*) .\end{aligned}$$

Set $\tilde{\mathbb{G}}_0 = \mathbb{G}_0(1 - x\tilde{E}_1)$, $\tilde{\mathbb{G}}_1 = \mathbb{G}_1(1 - a\tilde{E}_1)$, $\tilde{\mathbb{H}}_0 = \mathbb{G}_0(1 - a\tilde{E}_2)$, $\tilde{\mathbb{H}}_1 = \mathbb{G}_1(1 - a\tilde{E}_2)$, using (3.15),

$$\begin{aligned}\partial_{\mathcal{S}}F_N &= a^{3/2}r_c(\mathcal{S}^2 + Xq^{1/3}(1 + \ell) - \tilde{\mathbb{G}}_0 - \tilde{\mathbb{G}}_1 + \tilde{g}_1\lambda^{-2} + O_2 + a^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*) \\ \partial_{\mathcal{T}}F_N &= a^{3/2}r_c(\mathcal{T}^2 + q^{1/3} - \tilde{\mathbb{H}}_0 - \tilde{\mathbb{H}}_1 + \tilde{g}_2\lambda^{-2} + O_2 + a^{1/2}\partial_{\mathcal{T}}\tilde{\Psi}_*) .\end{aligned}$$

At critical points in \mathcal{S} of F_N , we have

$$\mathcal{S}^2 + Xq^{1/3}(1 + \ell) - \mathcal{A}_c(1 - \mathfrak{x}\tilde{E}_1) = O(a^{1/2})$$

as $(|\mathcal{S}|, |\mathcal{T}|) \leq M_0$ and

$$\partial_{\mathcal{S}}\tilde{\Psi}_* = X\mathcal{S}f + a^{1/2}X^2f + a^{1/2}\mathcal{A}Xf + a^{1/2}f, \quad \partial_{\mathcal{T}}\tilde{\Psi}_* = \mathcal{T}f + a^{1/2}f.$$

We may compute the second derivative with respect to \mathcal{S} ,

$$\begin{aligned}\partial_{\mathcal{S}}^2F_N &= a^{3/2}\partial_{\mathcal{S}}r_c(\mathcal{S}^2 + Xq^{1/3}(1 + \ell) - \mathcal{A}_c(1 - \mathfrak{x}\tilde{E}_1) + a^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*) \\ &\quad + a^{3/2}r_c(2\mathcal{S} - X\partial_{\omega}(1 + \ell)q^{1/3}(\omega_c)\partial_{\mathcal{S}}\omega_c \\ &\quad - \partial_{\mathcal{S}}\mathcal{A}_c(1 - \mathfrak{x}\tilde{E}_1) + \mathcal{A}_c\partial_{\omega}(\mathfrak{x}\tilde{E}_1)\partial_{\mathcal{S}}\omega_c) + a^2r_c\partial_{\mathcal{S}}^2\tilde{\Psi}_* .\end{aligned}$$

On the other hand, one can easily check from the previous steps that

$$\partial_{\mathcal{S}}r_c = O(N^{-1}), \quad \partial_{\mathcal{S}}\mathcal{A}_c = O(N^{-1}), \quad \partial_{\mathcal{S}}\omega_c = O(N^{-1}),$$

and we get that at critical points in $(\mathcal{S}, \mathcal{T})$ of F_N ,

$$(3.19) \quad \partial_{\mathcal{S}}^2F_N = a^{3/2}r_c(2\mathcal{S} + O(N^{-1}))$$

$$(3.20) \quad \partial_{\mathcal{T}}^2F_N = a^{3/2}r_c(2\mathcal{T} + O(N^{-1}))$$

$$\partial_{\mathcal{S}, \mathcal{T}}^2F_N = O(N^{-1}).$$

We now define $(\mathfrak{A}, \mathfrak{B})$,

$$-\mathfrak{A}\lambda^{-2/3} = Xq^{1/3}(1 + \ell) - \tilde{\mathbb{G}}_0, \quad -\mathfrak{B}\lambda^{-2/3} = q^{1/3} - \tilde{\mathbb{H}}_0$$

set $(\mathfrak{A}_0, \mathfrak{B}_0) = (\mathfrak{A}, \mathfrak{B})|_{s=\varrho=0}$ and $(\mathfrak{A}_0, \mathfrak{B}_0) = \mathfrak{r}(\cos \mathfrak{s}, \sin \mathfrak{s})$. Then rewrite

$$\begin{aligned}\partial_{\mathcal{S}}F_N &= a^{3/2}r_c(\mathcal{S}^2 - \mathfrak{A}\lambda^{-2/3} - \tilde{\mathbb{G}}_1 + O_2 + \tilde{g}_1\lambda^{-2} + a^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*) \\ \partial_{\mathcal{T}}F_N &= a^{3/2}r_c(\mathcal{T}^2 - \mathfrak{B}\lambda^{-2/3} - \tilde{\mathbb{H}}_1 + O_2 + \tilde{g}_2\lambda^{-2} + a^{1/2}\partial_{\mathcal{T}}\tilde{\Psi}_*)\end{aligned}$$

therefore, setting $G = h^{-1}F_N$, as

$$\partial_{\mathbf{y}}G = h^{-1}\lambda^{-1/3}\partial_{\mathcal{S}}F_N = \lambda^{2/3}r_c(\lambda^{-2/3}\mathbf{y}^2 - \mathfrak{A}\lambda^{-2/3} - \tilde{\mathbb{G}}_1 + O_2 + \tilde{g}_1\lambda^{-2} + a^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*)$$

from $\lambda^{2/3}\tilde{\mathbb{G}}_1 = O(\mathbf{x}, \mathbf{y})$, as well as

$$\begin{aligned}\lambda^{2/3}a^{1/2}\partial_{\mathcal{S}}\Psi_* &= \lambda^{2/3}a^{1/2}(X\mathcal{S}f + a^{1/2}X^2f + a^{1/2}\mathcal{A}Xf + a^{1/2}f) \\ &= O(\mathbf{y}) + O(1) \quad \text{when } |N| > \lambda^{1/3}\end{aligned}$$

which follows from $\lambda^{2/3}a^{1/2}X\mathcal{S}f = (\lambda^{2/3}/N)(a^{1/2}N)\lambda^{-1/3}\mathbf{y}Xf = O(\mathbf{y})$, as well as $\lambda^{2/3}af = (Na^{1/2})^2N^{-2}\lambda^{2/3}f = O(1)$, etc...

We get

$$\begin{aligned}\partial_{\mathbf{y}}G &= r_c(\mathbf{y}^2 - \mathfrak{A}_0 + O(\mathbf{x}, \mathbf{y}) + O(1)) \\ \partial_{\mathbf{x}}G &= r_c(\mathbf{x}^2 - \mathfrak{B}_0 + O(\mathbf{x}, \mathbf{y}) + O(1))\end{aligned}$$

Now, going back to (3.18), the integral is bounded for $0 \leq \mathbf{r} \leq \mathbf{r}_0$, by integration by parts for large (\mathbf{x}, \mathbf{y}) (recall the support of the integrand is now of radius $\lambda^{1/3}$).

For $\mathbf{r}_0 < r \leq c_0\lambda^{2/3}$, we rescale again, $(\mathbf{x}, \mathbf{y}) = r^{1/2}(\mathbf{x}', \mathbf{y}')$ and set $G = r^{3/2}G'$, $\chi'(\mathbf{x}', \mathbf{y}') = \underline{\chi}(r^{1/2}\lambda^{-1/3}\mathbf{x}', r^{1/2}\lambda^{-1/3}\mathbf{y}')$ and as $r^{1/2}\lambda^{-1/3}$ is bounded, we retain the decay

$$|\partial_{(\mathbf{x}', \mathbf{y}')}^\mu \chi'(\mathbf{x}', \mathbf{y}')| \leq C_\mu(1 + |\mathbf{x}'| + |\mathbf{y}'|)^{-|\mu|}.$$

We have to prove

$$|\int e^{ir^{3/2}G'} \chi' d\mathbf{x}' d\mathbf{y}'| \leq \frac{C}{r}.$$

To begin with, notice that

$$\begin{aligned}\partial_{\mathbf{y}'}G' &= r_c(\mathbf{y}'^2 - \sin \mathfrak{s} + r^{-1/2}O(\mathbf{x}', \mathbf{y}') + r^{-1}O(1)) \\ \partial_{\mathbf{x}'}G' &= r_c(\mathbf{x}'^2 - \cos \mathfrak{s} + r^{-1/2}O(\mathbf{x}', \mathbf{y}') + r^{-1}O(1))\end{aligned}$$

and from (3.19), we have

$$\partial_{\mathbf{y}}^2 G = \lambda^{1/3}r_c(2\lambda^{-1/3}\mathbf{y} + O(N^{-1})) = r_c(2\mathbf{y} + O(1))$$

where we used $|N| > \lambda^{1/3}$ in the last step. If $|\sin \mathfrak{s}| \geq \varepsilon_0 > 0$, G' will have two critical points in \mathbf{y}' , namely $\mathbf{y}'_{\pm} = \pm|\sin \mathfrak{s}|^{1/2} + O(r^{-1/2})$, and these critical points are non degenerate as $|\partial_{\mathbf{y}}^2 G| \geq c_0r^{1/2}$, which translates into $|\partial_{\mathbf{y}'}^2 G'| \geq c_0 > 0$. By stationnary phase, we get

$$r \int e^{ir^{3/2}G'} \chi' d\mathbf{x}' d\mathbf{y}' = r^{1/4} \left(\int e^{ir^{3/2}G'_+} \chi'_+ d\mathbf{x}' + \int e^{ir^{3/2}G'_-} \chi'_- d\mathbf{x}' \right).$$

From (3.20), one may check that $|\partial_{\mathbf{x}}^3 G'| \geq c_0$, and by degenerate stationnary phase we get

$$|\int e^{ir^{3/2}G'_-} \chi'_- d\mathbf{x}'| \leq C(r^{3/2})^{-1/3} = Cr^{-1/2},$$

and for $r \geq 1$, we get the desired decay, and even an extra $r^{-1/4}$ on the righthandside. When $|\cos \mathfrak{s}| \geq \varepsilon_0 > 0$, we proceed in the same way, exchanging \mathbf{x}' and \mathbf{y}' .

We proceed with the case $2 \leq |N| \leq \lambda^{1/3}$ (where we do not perform stationary phase in r !). Define (\mathbf{p}, \mathbf{q}) as

$$\begin{aligned} -\frac{\mathbf{p}}{N^2} &= Xq^{1/3}(1 + \ell) - \tilde{\mathbb{G}}_0 \\ -\frac{\mathbf{q}}{N^2} &= q^{1/3} - \tilde{\mathbb{H}}_0 \end{aligned}$$

and rescale variables with $(\mathbf{x}, \mathbf{y}) = -(\mathcal{T}|N|, -\mathcal{S}|N|)$. We will prove

Lemma 3.11. *For $|N| \geq 2$, set $\Lambda = \lambda|N|^{-3}$ and assume $\Lambda \geq 1$, we have*

$$(3.21) \quad \left| \int e^{\frac{i}{h}F_N} \chi\left(\frac{\mathbf{x}}{|N|}, \frac{\mathbf{y}}{|N|}\right) d\mathbf{x}d\mathbf{y} \right| \leq C\Lambda^{-3/4}.$$

Set $G_a = (h\Lambda)^{-1}F_N = |N|^3a^{-3/2}F_N$ and $\partial_{\mathbf{y}}G_a = -N^2a^{-3/2}\partial_{\mathcal{S}}F_N$, so that

$$\partial_{\mathbf{y}}G_a = N^2r\left(-\frac{\mathbf{y}^2}{N^2} + \frac{\mathbf{p}}{N^2} - \tilde{\mathbb{G}}_1 - \tilde{O}_2(1 - x\tilde{E}_1) + \frac{g_1}{\lambda^2}\right) - N^2ra^{1/2}\partial_{\mathcal{S}}\tilde{\Psi}_*$$

and

$$\partial_{\mathbf{x}}G_a = N^2r\left(-\frac{\mathbf{x}^2}{N^2} + \frac{\mathbf{q}}{N^2} - \tilde{\mathbb{H}}_1 - \tilde{O}_2(1 - a\tilde{E}_2) + \frac{\tilde{g}_1}{\lambda^2}\right) - N^2ra^{1/2}\partial_{\mathcal{T}}\tilde{\Psi}_*.$$

On the other hand, from $a^{1/2}N^2\partial_{\mathcal{S}}\Psi_* = a^{1/2}N^2(X\mathcal{S}f + a^{1/2}X^2f + a^{1/2}\mathcal{A}Xf + a^{1/2}f)$ and $a^{1/2}N^2\partial_{\mathcal{T}}\Psi_* = a^{1/2}N^2(\mathcal{T}f + a^{1/2}f)$ we can write

$$-\mathbb{G}_1 = \beta_1 \frac{\mathcal{S}}{2N} + \gamma_1 \frac{\mathcal{T}}{2N}$$

and from the previous definition of \mathbb{G}_1 , we get $\beta_1 = 2a_0b_0(T/(4N))$, $\gamma_1 = 2a_0c_0(T/(4N))$.

Moreover, $-N^2\tilde{\mathbb{G}}_1 = (\tilde{\beta}_1\mathbf{y} + \tilde{\gamma}_1\mathbf{x})$ with $\tilde{\beta}_1 = \beta_1 + O(a)$, $\tilde{\gamma}_1 = \gamma_1 + O(a)$. From $\tilde{O}_2 = (b_0\mathcal{S}/(2N) + c_0\mathcal{T}/(2N))^2 + a^{1/2}\tilde{O}'_2$ as well as

$$N^2\tilde{O}_2(1-x\tilde{E}_1) = \frac{(b_0\mathbf{y} + c_0\mathbf{x})^2}{4N^2} + a^{1/2}O((\mathbf{x}, \mathbf{y})^2), \quad N^2\tilde{O}_2(1-a\tilde{E}_2) = \frac{(b_0\mathbf{y} + c_0\mathbf{x})^2}{4N^2} + a^{1/2}O((\mathbf{x}, \mathbf{y})^2),$$

we can infer that

$$(3.22) \quad \partial_{\mathbf{y}}G_a = r(-\mathbf{y}^2 + \mathbf{p}_2 + \frac{1}{2}(\beta_2\mathbf{y} + \gamma_2\mathbf{x}) + \frac{(b_0\mathbf{y} + c_0\mathbf{x})^2}{4N^2} + (a^{1/2} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})^2)),$$

as well as

$$(3.23) \quad \partial_{\mathbf{x}}G_a = r(-\mathbf{x}^2 + \mathbf{q}_2 + \frac{1}{2}(\beta_3\mathbf{y} + \gamma_3\mathbf{x}) + \frac{(b_0\mathbf{y} + c_0\mathbf{x})^2}{4N^2} + (a^{1/2} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})^2)),$$

with $\beta_2, \beta_3 = \beta_1 + O(a^{1/2}N) + O(N\lambda^{-2})$, $\gamma_2 = \gamma_3 = \gamma_1 + O(a^{1/2}N) + O(N\lambda^{-2})$, $\beta_3 = \beta_1$, and $\mathbf{p}_2 = \mathbf{p} + O(\sqrt{a}N) + O(N\lambda^{-2})$, $\mathbf{q}_2 = \mathbf{q} + O(\sqrt{a}N) + O(N\lambda^{-2})$. Next, we compute

$$\partial_{\mathbf{y}}^2G_a = r(-2\mathbf{y} + \frac{1}{2}\beta_2 + \frac{2(b_0\mathbf{y} + c_0\mathbf{x})}{4N^2} + (a^{1/2} + \lambda^{-2})O((\mathbf{x}, \mathbf{y}))),$$

$$\partial_{\mathbf{x}}^2 G_a = r(-2\mathbf{x} + \frac{1}{2}\gamma_3 + \frac{2(b_0\mathbf{y} + c_0\mathbf{x})}{4N^2} + (a^{1/2} + \lambda^{-2})O((\mathbf{x}, \mathbf{y}))),$$

and

$$\partial_{\mathbf{x}, \mathbf{y}}^2 G_a = r(\frac{1}{2}\beta_3 + \frac{2(b_0\mathbf{y} + c_0\mathbf{x})}{4N^2} + (a^{1/2} + \lambda^{-2})O((\mathbf{x}, \mathbf{y}))).$$

from which we compute the determinant of the Hessian of G_a ,

$$H_a = r^2(4\mathbf{x}\mathbf{y} - (\beta_2\mathbf{x} + \gamma_3\mathbf{y}) + (\mathbf{x} + \mathbf{y})^2\lambda_0^2N^{-2} + O(a^{1/2}(\mathbf{x}, \mathbf{y})) + \beta_2\gamma_2 - \beta_3\gamma_2),$$

with $\beta_2 = 2\lambda_0(T/(4N))a_0(0) + O(a^{1/2}N)$ where $b_0(0) = c_0(0) = \lambda_0$, and $\gamma_3 = 2\lambda_0(T/(4N))a_0(0) + O(a^{1/2}N)$ (recall $b_0 = c_0 + O(a)$). Set $\mu_0 = a_0(0)b_0(0)$,

$$H_a = r^2(4\mathbf{x}\mathbf{y} - 2\mu_0(T/(4N))(\mathbf{x} + \mathbf{y}) + (\mathbf{x} + \mathbf{y})^2\lambda_0^2N^{-2} + O(N(a^{1/2} + \lambda^{-2})(\mathbf{x}, \mathbf{y})) + O((a^{1/2} + \lambda^{-2})N)).$$

Notice that in the model case (see [8]) $H_{a=0} = 4r^2(\mathbf{x}\mathbf{y} - \tilde{T}(\mathbf{x} + \mathbf{y})/2 - (\mathbf{x} + \mathbf{y})^2/(4N^2))$.

Set $(\mathbf{q}_2, \mathbf{p}_2) = \mathbf{r}(\cos \mathfrak{s}, \sin \mathfrak{s})$. We will again deal separately with $\mathfrak{r} \geq \mathfrak{r}_0$, with large \mathfrak{r}_0 , and $\mathfrak{r} < \mathfrak{r}_0$. We start with large \mathfrak{r} and will prove

$$\left| \int e^{i\Lambda G_a} \underline{\chi}\left(\frac{\mathbf{x}}{|N|}, \frac{\mathbf{y}}{|N|}\right) d\mathbf{x}d\mathbf{y} \right| \leq C\Lambda^{-5/6},$$

which has better decay than required. First, observe that from the hypothesis on \mathfrak{r} and (3.22), (3.23), we may integrate by parts in (\mathbf{x}, \mathbf{y}) in a region $|(\mathbf{x}, \mathbf{y})| \leq c\mathfrak{r}^{-1/2}$: for any $k \geq 1$,

$$\left| \int_{|(\mathbf{x}, \mathbf{y})| \leq c\mathfrak{r}^{-1/2}} e^{i\Lambda G_a} \underline{\chi}\left(\frac{\mathbf{x}}{|N|}, \frac{\mathbf{y}}{|N|}\right) d\mathbf{x}d\mathbf{y} \right| \leq C_k(\mathfrak{r}\Lambda)^{-k}.$$

Thus we are left with the region $|(\mathbf{x}, \mathbf{y})| \geq c\mathfrak{r}^{-1/2}$. We rescale again $(\mathbf{x}', \mathbf{y}') = \mathfrak{r}^{1/2}(\mathbf{x}, \mathbf{y})$ and define $G'_a = r^{3/2}G_a$ and prove

$$(3.24) \quad \left| \mathfrak{r} \int_{|(\mathbf{x}', \mathbf{y}')| \geq c} e^{i\Lambda r^{3/2}G'_a} \underline{\chi}\left(\frac{\mathfrak{r}^{1/2}\mathbf{x}'}{|N|}, \frac{\mathfrak{r}^{1/2}\mathbf{y}'}{|N|}\right) d\mathbf{x}'d\mathbf{y}' \right| \leq C\Lambda^{-5/6}.$$

We compute

$$\begin{aligned} \partial_{\mathbf{y}'} G'_a &= -r(\mathbf{y}'^2 - \sin \mathfrak{s} - \frac{(c_0\mathbf{x}' + b_0\mathbf{y}')^2}{4N^2} + O(a^{1/2}\mathfrak{r}^{-1/2})) \\ \partial_{\mathbf{x}'} G'_a &= -r(\mathbf{x}'^2 - \cos \mathfrak{s} - \frac{(c_0\mathbf{x}' + b_0\mathbf{y}')^2}{4N^2} + O(a^{1/2}\mathfrak{r}^{-1/2})). \end{aligned}$$

From these expressions of $\nabla G'_a$, if $(\mathbf{x}', \mathbf{y}')$ is very large, we get decay by integrations by parts, as $|N| \geq 2$ and $b_0 = c_0 = 1$ when $a = x = y = 0$. Therefore, we can assume that $c \leq |(\mathbf{x}', \mathbf{y}')| \leq C$ where C is a large fixed constant.

Let us denote by H'_a the determinant of the Hessian of G'_a , and $\Gamma = \{H'_a = 0\}$. If $|N| \geq 2$, a is small and \mathfrak{r}_0 is large, Γ is a smooth curve which does not intersect the origin. Away

from Γ , we may use stationary phase, which will provide $\mathfrak{r}(\mathfrak{r}^{3/2}\Lambda)^{-1}$ decay on the left of (3.24). In the region close to Γ , we need to apply Lemma 2.21 (a) in [8], e.g. degenerate stationary phase along a curve, to get

$$\left| \int_{c \leq |(\mathbf{x}', \mathbf{y}')| \leq C} e^{i\Lambda r^{3/2} G'_a} \underline{\chi} \left(\frac{r^{1/2} \mathbf{x}'}{|N|}, \frac{r^{1/2} \mathbf{y}'}{|N|} \right) d\mathbf{x}' d\mathbf{y}' \right| \leq C(\mathfrak{r}^{3/2}\Lambda)^{-5/6},$$

and therefore we get (3.24) as the extra factor \mathfrak{r} on the lefthand side is cancelled by the $\mathfrak{r}^{-5/4}$ on the righthand side (recall $\mathfrak{r} \geq \mathfrak{r}_0$).

We can now focus on $|(\mathbf{p}, \mathbf{q})| \leq r_0$. Notice that there exists $c_0 > 0$ such that

$$|\mathbf{x}^2 - \frac{(c_0 \mathbf{x} + b_0 \mathbf{y})^2}{4N^2}|^2 + |\mathbf{y}^2 - \frac{(c_0 \mathbf{x} + b_0 \mathbf{y})^2}{4N^2}|^2 \geq c_0(\mathbf{x}^2 + \mathbf{y}^2)$$

at least when a is small and for all $|N| \geq 2$, as it holds for $a = x = y = 0$ and $|(\mathbf{x}, \mathbf{y})| = 1$.

Therefore, again, we may reduce to $|(\mathbf{x}, \mathbf{y})| \leq R$ with large R as we get $\Lambda^{-\infty}$ decay by integration by parts if $|(\mathbf{x}, \mathbf{y})|$ is large, as we did before. We now aim at proving

$$(3.25) \quad \left| \int_{|(\mathbf{x}, \mathbf{y})| \leq R} e^{i\Lambda G_a} \underline{\chi} d\mathbf{x} d\mathbf{y} \right| \leq C\Lambda^{-3/4},$$

for which we will apply part (b) of Lemma 2.21 in [8]. Recall H_a is the determinant of the Hessian of G_a , and

$$H_a = r^2(4\mathbf{x}\mathbf{y} - 2\mu_0(T/(4N))(\mathbf{x} + \mathbf{y}) - \lambda_0^2 N^{-2}(\mathbf{x} + \mathbf{y})^2 + O(a^{1/2}N))$$

and denote $Z_a = \{H_a = 0\}$; Z_a is a smooth curve in $B = \{|(\mathbf{x}, \mathbf{y})| \leq R\}$, at least for small a : it will be close to the parabola $-2(T/(4N))(\mathbf{x} + \mathbf{y}) = (\mathbf{x} - \mathbf{y})^2$ when $|N| = 1$ and to the hyperbola $-2(T/(4N))(\mathbf{x} + \mathbf{y}) + 4\mathbf{x}\mathbf{y} = (\mathbf{x} + \mathbf{y})^2/N^2$ for $|N| \geq 2$ (note that $\mu_0 = \lambda_0 = 1$ when $(t, x, y) = 0$).

One has

$$\begin{aligned} \partial_{\mathbf{x}} G_a &= r(-\mathbf{x}^2 + q_2 + \frac{1}{2}(\beta_3 \mathbf{y} + \gamma_3 \mathbf{x}) + \frac{(b_0 \mathbf{y} + c_0 \mathbf{x})^2}{4N^2} + (\sqrt{a} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})^2)) \\ \partial_{\mathbf{y}} G_a &= r(-\mathbf{y}^2 + p_2 + \frac{1}{2}(\beta_2 \mathbf{y} + \gamma_2 \mathbf{x}) + \frac{(b_0 \mathbf{y} + c_0 \mathbf{x})^2}{4N^2} + (\sqrt{a} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})^2)), \end{aligned}$$

and in the ball B we have $O((\mathbf{x}, \mathbf{y})^2) = O(1)$. We set

$$\begin{aligned} \alpha &= r^{-1} \partial_{\mathbf{x}}^2 G_a = -2\mathbf{x} + \frac{1}{2}\gamma_3 + c_0 \frac{(b_0 \mathbf{y} + c_0 \mathbf{x})}{2N^2} + (\sqrt{a} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})) \\ \beta &= r^{-1} \partial_{\mathbf{x}\mathbf{y}}^2 G_a = \frac{1}{2}(\beta_3 + b_0 \frac{(b_0 \mathbf{y} + c_0 \mathbf{x})}{2N^2}) + (\sqrt{a} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})) \\ \gamma &= r^{-1} \partial_{\mathbf{y}}^2 G_a = -2\mathbf{y} + \frac{1}{2}\beta_2 + b_0 \frac{(b_0 \mathbf{y} + c_0 \mathbf{x})}{2N^2} + (\sqrt{a} + \lambda^{-2})O((\mathbf{x}, \mathbf{y})) \end{aligned}$$

so that the Hessian of G_a is $r \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = rM(\mathbf{x}, \mathbf{y})$. Let $\xi_0 = (\mathbf{x}_0, \mathbf{y}_0) \in B$, if $H_a(\xi_0) \neq 0$, then the usual stationary phase applies, unless we are in the special condition where Lemma 2.21 in [8] takes over. Let $H(\xi) = G_a(\xi + \xi_0) - G_a(\xi_0) - \nabla G_a(\xi_0) \cdot \xi$: we see that $H(0) = 0$, $H'(0) = 0$ and $H''(0) = \nabla^2 G_a(\xi_0)$. Let $H_a = \det H''(\xi)$. Assume $H_a(\xi_0) = 0$ and set $w = \nabla \det H''(0)$.

The matrix M has two eigenvalues, 0 with normalized eigenvector $v(\xi_0)$ and $\text{tr}M(\xi_0)$ with normalized eigenvector $u(\xi_0)$. Let Q be the matrix formed with column vectors u and v , then ${}^tQM(\xi_0)Q = \text{diag}(\lambda_1/2, 0)$, and $\lambda_1 \neq 0$ as the rank of $M(\xi_0) \geq 1$ (thanks to $|N| \geq 2$ and (t, x, y) close to 0). A simple computation yields

$$\frac{1}{2}|w|^2 = -2(-2 + \lambda_0^2 N^{-2})(\mathbf{x}_0 - \mathbf{y}_0)^2 + \mu_0^2(T/(4N))^2 + O(a^{1/2}N) + O(N\lambda^{-2})$$

and therefore $|w| \geq c_0 > 0$ provided $a^{1/2}N$ is small enough. Denote by $H_1(\eta) = H(Q\eta)$, we have

$$(3.31) \quad H_1(\xi) = \lambda_1(\xi_1^2/2 + a\xi_1^3 + b\xi_1^2\xi_2 + c\xi_1\xi_2^2 + d\xi_2^3 + e\xi_2^4 + \dots + O(\xi^5))$$

and $\det H_1''(\xi) = \lambda_1^2(2c\xi_1 + 6d\xi_2) + O(\xi^2)$, so $|c| + |d| \geq c_0 > 0$ while, if $X(s)$ is a solution to $X(s) = H'(\xi(s))$ (and $\xi(s)$ is a smooth parametrization of Γ_a near $a = 0$), $X'(0) \neq 0$ yields $d \neq 0$ and $X'(0) = 0$, $X''(0) \neq 0$ yields $d = 0$, $c \neq 0$ and $c^2 \neq 2e$.

Write

$$M(\xi_0 + \xi) = M(\xi_0) + M_1\xi_1 + M_2\xi_2 + \sum_{i,j=1,2} M_{ij}\xi_i\xi_j + O(|\xi|^3)$$

then $M_{2,2} = (\partial_y^2 M)(\xi_0)$ is small, as α, β, γ are linear in (\mathbf{x}, \mathbf{y}) up to $(\sqrt{a} + \lambda^{-2})O(\mathbf{x}, \mathbf{y})$ remainders. Here $O(\mathbf{x}, \mathbf{y})$ means functions like $p\mathbf{x} + q\mathbf{y}$ where p, q are smooth functions of $(N^{-1}\mathbf{x}, N^{-1}\mathbf{y})$, and $a^{1/2}\partial_y^2 O(\mathbf{x}, \mathbf{y}) = O(a^{1/2}N^{-1})$. As such, the e coefficient in the expansion (3.31) is $O((\sqrt{a} + \lambda^{-2})N^{-1})$, we have $|c| + |d| \geq c_0 > 0$ at least for small a and (t, x, y) close to zero, and finally, $d^2 + |c^2 - 2e| \geq c_0^2 - O(a^{1/2} + \lambda^{-2}) \geq c_0^2/2$. Following [8] we get the desired bound (3.25). \square

We briefly digress to provide a brief outline of the proof of Theorem 1.3. This reduces to proving that the bound we just proved, e.g. (3.25), is optimal: indeed, as in [8], the symbol χ is elliptic at $(0, 0)$, we may use the second bound in part b) of Lemma 2.21 in [8] and for each N , each Θ , we have a time $T_{N,\Theta} \sim 4N(1 + O(a, \Theta))$ and points $(X = 1, Y_{N,\Theta})$ at which (3.25) may be reversed, proving the desired optimality. \square

We now proceed with the estimate on the sum over N , e.g.

$$\left| \sum_{|N| \geq 2} V_{N,2}(x, y, t) \right| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} (a^{1/4} \left(\frac{h}{t} \right)^{\frac{1}{4}} + h^{1/3})$$

provided $h \in (0, h_0]$, $a \in [h^\kappa, a_0]$ with $\kappa = 2/3 - \varepsilon$, and small $\varepsilon > 0$.

We start with a few preliminary remarks

- if $N \notin \mathcal{N}_1(x, y, t)$, we may just integrate by parts to get $O_{C^\infty}(h^\infty)$;
- if $\beta = (\omega, r, \mathcal{A}, \mathcal{S}, \mathcal{T})$ is a critical point in the phase integral defining $V_{N,2}$, then $T/|N| \equiv \mathcal{A}^{1/2}$;
- if $N \in \mathcal{N}_1(x, y, t)$, there exists $(x', y', t') \in B_0(x, y, t)$ such that $N \in \mathcal{N}(x', y', t')$, so that there exists \mathcal{T} such that $\beta' = (\omega, r, \mathcal{A}, \mathcal{S}, \mathcal{T})$ is a critical point of F_N and $T'/|N| \equiv \mathcal{A}^{1/2}$ as $|T - T'| \leq \varepsilon_0$ and $T \geq T_0 > 0$, $T/|N| \equiv \mathcal{A}^{1/2}$. Therefore on $|\mathcal{A}| \leq$

\mathcal{A}'_0 , we obviously have $|N| \sim T$, and if not, we have $|\mathcal{A}| \geq |\mathcal{A}'_0|$: this implies then that $\nabla_{\mathcal{A}} F_N \neq 0$ on the support of $q_{h,2}$, and uniformly in N , $V_{N,2} \in O(h^\infty |N|^{-\infty})$.

Set $\Delta N = |\mathcal{N}_1(x, y, t)|$. If $\Delta N \leq 2C$ where C is some large constant, there is only a finite number of terms in the sum, discarding all $N \notin \mathcal{N}_1(x, y, t)$.

- If for those $N \in \mathcal{N}_1(x, y, t)$, we have $|N| \leq \lambda^{1/3}$, and collecting (3.10) (stationnary phase in ω), (3.13) (stationnary phase in \mathcal{A}) and (3.21) (degenerate stationnary phase in $(\mathcal{S}, \mathcal{T})$) we get

$$|V_{N,2}(x, y, t)| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \frac{a^2}{h} \frac{1}{|N|^{1/2} \lambda^{1/2}} |N|^{1/4} \lambda^{-3/4}$$

which reduces to

$$|V_{N,2}(x, y, t)| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \frac{a^{1/8} h^{1/4}}{|N|^{1/4}}$$

and from $t \sim a^{1/2} N$ (see (3.14)) we get the desired result.

- If we have $|N| \geq \lambda^{1/3}$, we collect the same bounds with (3.21) replaced by (3.24) and get

$$(3.32) \quad |V_{N,2}(x, y, t)| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \frac{1}{|N|^{1/2} \lambda^{1/2}} a^{1/2} \lambda^{1/3}$$

and one easily checks that $a^{1/2} N^{-1/2} \lambda^{-1/6} \leq a^{1/8} h^{1/4} N^{-1/4}$ is equivalent to $|N| \geq \lambda^{1/3}$ (we may also bound $1/N$ by $\lambda^{-1/3}$ to get $h^{1/3}$ instead of $a^{1/8} h^{1/4} / |N|^{1/4}$.)

Remark 3.12. The condition $|\mathcal{N}_1(x, y, t)| \leq 2C$ is related to the overlapping of $V_{N,2}$ when N is varying. Essentially, only a finite number of such $V_{N,2}$ are overlapping, and this corresponds to $a > h^{4/7}$.

We proceed with ΔN large, e.g. $\Delta N \geq 2C$: as $\Delta N \leq C + CT\lambda^{-2}$, this implies that $T \geq c\lambda^2$ as $N \in \mathcal{N}_1(x, y, t)$. As we noticed earlier in estimating $V_{N,2}$, we also have $|N| \sim T$ (see (3.14)).

For those N , obviously $|N| \geq \lambda^{1/3}$, and

$$\left| \sum_{N \in \mathcal{N}_1(x, y, t), |N| \sim T} V_{N,2} \right| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} a^{1/2} \lambda^{1/3} \frac{1}{|N|} \Delta N$$

We note that we improved a factor $N^{-1/2} \lambda^{-1/2}$ in (3.32) into N^{-1} by taking advantage of the additional gain in (3.17), which is $(N\lambda^{-1})^{-1/2}$ (stationnary phase in r , which is the length of the Fourier variable associated to the tangential variable, see (3.16)).

Therefore,

$$\left| \sum_{N \in \mathcal{N}_1(x, y, t), |N| \sim T} V_{N,2} \right| \leq \frac{C_0}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} a^{1/2} \lambda^{-5/3},$$

and one easily checks that $a^{1/2} \lambda^{-5/3} \leq h^{1/3}$ in our range.

We now deal with the $N = 1$ case: one may inspect the previous proof to check that, knowing from support conditions on χ that (\mathbf{x}, \mathbf{y}) is bounded, for large (\mathbf{p}, \mathbf{q}) integrations by parts provide decay. On the other hand, in the range (\mathbf{p}, \mathbf{q}) bounded, one may proceed as before, using Lemma 2.22 part b).

3.2. The term $V_{N,1}$. Here, the main difference with the previous case is that the critical point A_c need not be bounded. Recall

$$V_{N,1} = \frac{a^2}{h^{d+1}} \int e^{\frac{i}{h} \Psi_N} q_{h,1} d\theta dA dS d\Upsilon,$$

where on the support of $q_{h,1}$, we have $A \geq A_1 > 0$ with A_1 a large fixed constant. We proceed in a different order and will perform stationnary phase in (S, Υ) : critical points are non degenerate, as we will see in a moment.

Recall

$$\Psi_N = r(\psi_0 + a^{3/2} \Psi_{N,0} + a^{3/2} EA + a^2 \Psi_*) ,$$

where

$$\psi_0 = y(\omega + \nu_0(y, \omega)) ,$$

$\Psi_{N,0}$ is essentially the model phase

$$\begin{aligned} \Psi_{N,0} = & s^3/3 + S(q^{1/3}(\omega)(1 + \ell)(y, \omega)X - A) + \Upsilon^3/3 + \Upsilon q^{1/3}(\omega) - \Upsilon A \\ & - \frac{4}{3}NA^{3/2} + NB(\lambda A^{3/2})\lambda^{-1} + \frac{1}{2}TAq^{2/3}(\omega) , \end{aligned}$$

and

$$\begin{aligned} E = & E_0(y, \omega)T + E_1(y, \omega)Sx + E_2(y, \omega)\Upsilon a , \\ \Psi_* = & XS^2f_1 + aXf_2 + a^{1/2}X^2Sf_3 + \Upsilon^2f_4 + a^{3/2}A^2\Upsilon f_5 + f_6 + Ta^{1/2}A^2f_7 . \end{aligned}$$

Write $\Theta_N = \Psi_{N,0} + EA$, then

$$\begin{aligned} \partial_S \Theta_N = & S^2 + Xq^{1/3}(1 + \ell) - A(1 - E_1x) \\ \partial_\Upsilon \Theta_N = & \Upsilon^2 + q^{1/3} - A(1 - E_2a) \end{aligned}$$

while

$$\begin{aligned}\partial_S \Psi_* &= X S f + a^{1/2} X^2 f + a^{1/2} f \\ \partial_\Upsilon \Psi_* &= \Upsilon f + a^{1/2} f.\end{aligned}$$

Therefore, if $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$, critical points of Θ_N are $(\varepsilon_1 S_0, \varepsilon_2 T'_0)$ are such that

$$\begin{aligned}S_0^2 &= A(1 - E_1 x) - X q^{1/3}(1 + \ell) \\ T_0'^2 &= A(1 - E_2 x) - X q^{1/3}.\end{aligned}$$

One should keep in mind that E_1 and E_2 are both $O(y)$.

Critical points of Ψ_N are then solutions to

$$\begin{aligned}S^2 &= S_0^2 - a^{1/2} \partial_S \Psi_* \\ T^2 &= T_0'^2 - a^{1/2} \partial_\Upsilon \Psi_*.\end{aligned}$$

One may check that $\partial_S^2 \Psi_*$, $\partial_\Upsilon^2 \Psi_*$ and $\partial_{S,\Upsilon}^2 \Psi_*$ are $O(1)$. On the other hand,

$$\begin{aligned}\partial_S^2 \Psi_N &= r a^{3/2} (2S + a^{1/2} \partial_S^2 \Psi_*) \\ \partial_\Upsilon^2 \Psi_N &= r a^{3/2} (2\Upsilon + a^{1/2} \partial_\Upsilon^2 \Psi_*) \\ \partial_{S,\Upsilon}^2 \Psi_N &= r a^{3/2} a^{1/2} \partial_{S,\Upsilon}^2 \Psi_*.\end{aligned}$$

Rescale variables $(S, \Upsilon) = k(\sigma, \tau')$ with $k = A^{1/2}$, one has

$$\begin{aligned}\partial_\sigma \Psi_N &= k^3 a^{3/2} (\sigma^2 - \sigma_0^2 + a^{1/2} k^{-1/2} \partial_S \Psi_*) \\ \partial_{\tau'} \Psi_N &= k^3 a^{3/2} (\tau'^2 - \tau_0'^2 + a^{1/2} k^{-1/2} \partial_\Upsilon \Psi_*),\end{aligned}$$

with $\sigma_0^2 = (1 - x E_1) - A^{-1} X q^{1/3}(1 + \ell)$ and $\tau_0'^2 = (1 - a E_2) - q^{1/3}$. From

$$a^{1/2} (|\partial_S \Psi_*| + |\partial_\Upsilon \Psi_*|) \leq C \varepsilon_0, \quad \nabla_{S,\Upsilon}^2 \Psi_* = O(1),$$

one may apply the implicit function theorem at $(\varepsilon_1 \sigma_0, \varepsilon_2 \tau_0')$ and $a^{1/2} = 0$. The critical points in (σ, τ') are non degenerate, as

$$\nabla_{\sigma, \tau'}^2 \Psi_N = k^3 (a^{3/2} \begin{pmatrix} 2\sigma & 0 \\ 0 & 2\tau' \end{pmatrix} + O(a^2))$$

and we take $\mu = k^3/h$ as a new large parameter with which to apply the usual sationnary phase. Therefore,

$$V_{N,1} = \sum_{\varepsilon_1, \varepsilon_2} \int e^{\frac{i}{h} \Psi_{N, \varepsilon_1, \varepsilon_2}} \frac{a^{1/2}}{h^d} q_{h, \varepsilon_1, \varepsilon_2}^1 d\theta dA + O_{C^\infty}(h^\infty),$$

with $\Psi_{N, \varepsilon_1, \varepsilon_2} = \Psi_N|_{S=S_{\varepsilon_1}, \Upsilon=\Upsilon_{\varepsilon_2}}$, and $q_{h, \varepsilon_1, \varepsilon_2}^1$ is a symbol that is such that, uniformly in its parameters,

$$|\partial_\theta^\kappa \partial_A^l q_{h, \varepsilon_1, \varepsilon_2}^1| \leq C_{\kappa, l} A^{-1/2-l}.$$

Now, we change variable A to $\Gamma = A^{3/2}$ and get

$$(3.33) \quad V_{N,1} = \sum_{\varepsilon_1, \varepsilon_2} \int e^{\frac{i}{h} \Psi_{N, \varepsilon_1, \varepsilon_2}} \frac{a^{1/2}}{h^d} q_{h, \varepsilon_1, \varepsilon_2} d\theta d\Gamma + O_{C^\infty}(h^\infty),$$

$$\begin{aligned} \Psi_{N, \varepsilon_1, \varepsilon_2} = & r\{\psi_0 + a^{3/2} \Psi_{N,0}|_{S=S_{\varepsilon_1}, \Upsilon=\Upsilon_{\varepsilon_2}} \\ & + a^{3/2} (E_0 T + E_1 x S_{\varepsilon_1} + a E_2 T'_{\varepsilon_2})\} + a^2 \Psi_{*|S=S_{\varepsilon_1}, \Upsilon=\Upsilon_{\varepsilon_2}}, \end{aligned}$$

and $q_{h, \varepsilon_1, \varepsilon_2}$ is such that

$$|\partial_\theta^\kappa \partial_\Gamma^l q_{h, \varepsilon_1, \varepsilon_2}^1| \leq C_{\kappa, l} \Gamma^{-2/3-l}.$$

We now study critical points in Γ of the phase function $\Psi_{N, \varepsilon_1, \varepsilon_2}$. As in [8], the most important case is $T \geq T_0 > 0$ and $|N| \leq C_1 T$, with a large constant C_1 . First, one may check that, for $\Gamma \geq \Gamma_0 > 0$, $T \geq T_0$ with T_0 large and $|N| \leq C_0 a^{-1/2}$,

$$|\partial_\Gamma^2 \Psi_{N, \varepsilon_1, \varepsilon_2}| \geq c_0 T \Gamma^{-4/3} a^{3/2},$$

which guarantees that no critical point, if any, will be degenerate.

First, we check that for $T \geq T_0$ and $\Gamma \geq \Gamma_0$, we have

$$|S_{\varepsilon_1}| + |\Upsilon_{\varepsilon_2}| \leq C \Gamma^{1/3}.$$

Moreover, one has

$$(\nabla_{S, \Upsilon}^2 \Psi_{N, \varepsilon_1, \varepsilon_2}) \cdot (\partial_\Gamma S_{\varepsilon_1}, \partial_\Gamma \Upsilon_{\varepsilon_2}) = -(\partial_{S, \Gamma}^2 \Psi_N, \partial_{\Upsilon, \Gamma}^2 \Psi_N),$$

and as observed earlier,

$$\|(\nabla_{S, \Upsilon}^2 \Psi_{N, \varepsilon_1, \varepsilon_2})^{-1}\| \leq C \Gamma^{-1/3} a^{-3/2} \quad \text{and} \quad \|\nabla_{(S, \Upsilon), \Gamma}^2 \Psi_N\| \leq C \Gamma^{-1/3} a^{3/2},$$

so that

$$|\nabla_\Gamma(S_{\varepsilon_1}, \Upsilon_{\varepsilon_2})| \leq C \Gamma^{-2/3}$$

which may be iterated to yield, for all multiindex κ ,

$$|\nabla_\Gamma^\kappa(S_{\varepsilon_1}, \Upsilon_{\varepsilon_2})| \leq C_\kappa \Gamma^{1/3-|\kappa|}.$$

From the asymptotics of the B function and its derivatives, e.g. $|B^{(k)}(U)| \leq C U^{-k-1}$ for $U \geq U_0 > 0$, we get, for any $k \geq 1$,

$$|\partial_\Gamma^k N B'(\lambda \Gamma)| \leq C T \lambda^{-2} \Gamma^{-2-k}$$

and notice that

$$|\partial_\Gamma N B'(\lambda \Gamma)| \leq C T \lambda^{-2} \Gamma^{-3} \leq C T \lambda^{-2} \Gamma^{-4/3}$$

as Γ is large. Moreover, as q is large, we also have

$$\partial_\Gamma q^{2/3}(\omega) \frac{T}{2} \Gamma^{-1/3} \geq c_0 T \Gamma^{-4/3}.$$

By direct computation, we have

$$\begin{aligned} \partial_\Gamma \Psi_{N, \varepsilon_1, \varepsilon_2} &= a^{3/2} \frac{2}{3} \Gamma^{-1/3} r(-(S_{\varepsilon_1} + \Upsilon_{\varepsilon_2}) - 2N\Gamma^{1/3}(1 - \frac{3}{4}B'(\lambda\Gamma)) + \frac{T}{2}q^{2/3}(\omega) \\ &\quad + E_0T + E_1Sx + E_2a\Upsilon) \\ &\quad + a^2(t\Gamma^{1/3}f + af\Gamma^{-1/3} + a^{3/2}\Upsilon\Gamma^{1/3}f \\ &\quad + a\Upsilon^2\Gamma^{-1/3}f + a^{3/2}X^2S\Gamma^{-1/3}f + aS^2X\Gamma^{-1/3}f + \Gamma^{-1/3}Xf). \end{aligned}$$

Denote by $a^2\mathfrak{N}$ the last line in the previous equation: we will prove that

$$|\partial_\Gamma(a^{1/2}\mathfrak{N})| \leq C\varepsilon_0 T \Gamma^{-4/3}$$

which then easily yields that, for large $T \geq T_0$ and $\lambda \geq \lambda_0$, with small $\varepsilon_0 > 0$,

$$|a^{-3/2}\partial_\Gamma^2\Psi_{N, \varepsilon_1, \varepsilon_2}| \geq c_0 T \Gamma^{-4/3}.$$

There are seven terms to check in \mathfrak{N} . Start with

$$\begin{aligned} a^{1/2}t\partial_\Gamma(\Gamma^{1/3}f) &= aT\partial_\Gamma(\Gamma^{1/3}f) \\ &= aT(\Gamma^{-2/3}f + \Gamma^{1/3}a\Gamma^{-1/3}f) \\ &= aT(\Gamma^{-2/3}f + af) \end{aligned}$$

and from $|Ta\Gamma^{-2/3}f| \leq CT\Gamma^{-4/3}(a\Gamma)^{2/3} \leq CT(aA)\Gamma^{-4/3} \leq C\varepsilon_0 T\Gamma^{-4/3}$ and $|Ta^2f| \leq CT\Gamma^{-4/3}a^2\Gamma^{4/3} \leq C\varepsilon_0 T\Gamma^{-4/3}$, we get

$$|a^{1/2}t\partial_\Gamma(\Gamma^{1/3}f)| \leq C\varepsilon_0 T \Gamma^{-4/3}.$$

We move to the second one:

$$a^{1/2}\partial_\Gamma(af\Gamma^{-1/3}) = a^{3/2}f\Gamma^{-4/3} + a^{3/2}af\Gamma^{-2/3}$$

and from

$$|a^{3/2}f\Gamma^{-4/3}| \leq C\varepsilon_0 \Gamma^{-4/3}, \quad a^{3/2}af\Gamma^{-2/3} = a^{3/2}\Gamma^{-4/3}(a\Gamma^{2/3}f),$$

we get $|a^{3/2}af\Gamma^{-2/3}| \leq C\varepsilon_0 \Gamma$ and we are done with the second term.

We move to the third one:

$$a^{1/2}\partial_\Gamma(a^{3/2}\Upsilon\Gamma^{1/3}f) = a^2\Upsilon\Gamma^{-2/3}f + a^2\partial_\Gamma\Upsilon\Gamma^{1/3}f + a^2\Upsilon\Gamma^{1/3}\partial_\Gamma f$$

and we write $a^2\Upsilon\Gamma^{-2/3}f = (a^{1/2}\Upsilon)\Gamma^{-4/3}a^{3/2}\Gamma^{2/3}f = (a^{1/2}\Upsilon)\Gamma^{-4/3}(aA)f$, so that

$$|a^2\Upsilon\Gamma^{-2/3}f| \leq C\varepsilon_0 \Gamma^{-4/3}a^{1/2};$$

Then we have

$$|a^2 \partial_\Gamma \Gamma^{1/3} f| \leq C a^2 \Gamma^{-2/3} \Gamma^{1/3} = C \Gamma^{-4/3} \Gamma a^2 = C \Gamma^{-4/3} (aA)^{3/2} a^{1/2} \leq C \varepsilon_0 \Gamma^{-4/3} a^{1/2};$$

finally,

$$\begin{aligned} |a^2 \Upsilon \Gamma^{1/3} \partial_\Gamma f| &\leq |a^2 \Upsilon \Gamma^{1/3} a f \Gamma^{-1/3}| \leq |a^3 \Upsilon f| \leq C \Gamma^{1/3} a^3 = C \Gamma^{-4/3} \Gamma^{5/3} a^3 \\ &\leq C \Gamma^{-4/3} (A^{5/2} a^3) \leq C (Aa)^{5/2} \Gamma^{-4/3} a^{1/2} \leq C \varepsilon_0 a^{1/2} \Gamma^{-4/3}, \end{aligned}$$

which closes the estimate on the third term.

Move on to the fourth,

$$a^{1/2} \partial_\Gamma (a \Upsilon^2 \Gamma^{-1/3} f) = a^{3/2} \Upsilon^2 \Gamma^{-4/3} f + a^{3/2} \Upsilon \partial_\Gamma \Upsilon \Gamma^{-1/3} f + a^{3/2} \Upsilon^2 \Gamma^{-1/3} f$$

we have

$$\begin{aligned} |a^{3/2} \Upsilon^2 \Gamma^{-4/3} f| &\leq C a^{3/2} \Gamma^{2/3} \Gamma^{-4/3} = C a^{1/2} (aA) \Gamma^{-4/3} \leq C \varepsilon_0 a^{1/2} \Gamma^{-4/3} \\ |a^{3/2} \Upsilon \partial_\Gamma \Upsilon \Gamma^{-1/3} f| &\leq C a^{3/2} \Gamma^{1/3} \Gamma^{-2/3} \Gamma^{-1/3} = C a^{1/2} (aA) \Gamma^{-4/3} \leq C \varepsilon_0 a^{1/2} \Gamma^{-4/3} \\ |a^{3/2} \Upsilon^2 \Gamma^{-1/3} \partial_\Gamma f| &= |a^{3/2} \Upsilon^2 \Gamma^{-1/3} a f \Gamma^{-1/3}| \leq C a^{3/2} (\Upsilon)^2 \Gamma^{-2/3} a \\ &\leq C a^2 = C \Gamma^{-4/3} (a^2 \Gamma^{4/3}) = C \Gamma^{-4/3} (aA)^2 \leq C \varepsilon_0 \Gamma^{-4/3} \end{aligned}$$

which is what we need.

The fifth term is easy,

$$|a^{1/2} \partial_\Gamma (a^{3/2} X^2 S \Gamma^{-1/3} f)| = |a^{1/2} X^2 \partial_\Gamma (a^{3/2} S \Gamma^{-1/3} f)| \leq C a^{1/2} \varepsilon_0 \Gamma^{-4/3}.$$

For the sixth term, we write

$$a^{1/2} \partial_\Gamma (X \Gamma^{-1/3} f) = a^{1/2} X (\Gamma^{-4/3} f + \Gamma^{-1/3} \partial_\Gamma f) = a^{1/2} X (\Gamma^{-4/3} f + a \Gamma^{-2/3} f)$$

and then we have

$$|a^{1/2} X \Gamma^{-4/3} f| \leq C a^{1/2} \Gamma^{-4/3} \leq C \varepsilon_0 \Gamma^{-4/3},$$

as well as

$$|a^{3/2} X \Gamma^{-2/3} f| \leq C a^{3/2} \Gamma^{-4/3} \Gamma^{2/3} \leq C \Gamma^{-4/3} (aA) a^{1/2} \leq C a^{1/2} \Gamma^{-4/3} \varepsilon_0 \Gamma^{-4/3},$$

which bounds our sixth term as well.

We are left with the last term, which is

$$a^{1/2} \partial_\Gamma (a S^2 X \Gamma^{-1/3} f) = X a^{3/2} \partial_\Gamma (S^2 \Gamma^{-1/3} f)$$

which is similar to a previous term from the 5th one. \square

Moreover, we can deduce from the above calculations that for Γ_0 large enough, there is at most one critical point Γ_c of $\Psi_{N, \varepsilon_1, \varepsilon_2}$ on the support of $q_{h, \varepsilon_1, \varepsilon_2}^1$ satisfying $\Gamma_c \sim (T/N)^3$, and of course this critical point is non degenerate. As S_{ε_1} and Υ_{ε_2} do not depend on r and

$\tilde{\Psi}_*$ is, by homogeneity, independent of r as well, Γ_c depends on r only through λ . Given that

$$\partial_\lambda \Gamma_c \partial_\Gamma^2 \Psi_{N,\varepsilon_1,\varepsilon_2}|_{\Gamma=\Gamma_c} = -\partial_\lambda \partial_\Gamma \Psi_{N,\varepsilon_1,\varepsilon_2}|_{\Gamma=\Gamma_c} = a^{3/2} N \Gamma_c B''(\lambda \Gamma_c),$$

we have $|\lambda \partial_\lambda \Gamma_c| \leq C \lambda^{-2} \Gamma_c^{-1}$, and iterating, we obtain that for all $k \geq 1$, $|(\lambda \partial_\lambda)^k \Gamma_c| \leq c_K$.

We proceed with estimates on derivatives of S_{ε_1} , Υ_{ε_2} and Γ_c with respect to ω .

First, we easily see from the previous arguments that

$$|\partial_\omega^\kappa \partial_\Gamma^l (S_{\varepsilon_1}, \Upsilon_{\varepsilon_2})| \leq C_{\kappa,l} \Gamma_c^{1/3-l} \quad \text{and} \quad |\partial_\omega^\kappa \partial_\Gamma^l \Gamma_c| \leq C_{\kappa,l} \Gamma_c.$$

Now define

$$\tilde{F}_{N,\varepsilon_1,\varepsilon_2} = r^{-1} F_{N,\varepsilon_1,\varepsilon_2} = r^{-1} \Psi_{N,\varepsilon_1,\varepsilon_2}|_{\Gamma=\Gamma_c} = \tilde{\Psi}_{N|S_{\varepsilon_1}, \Upsilon_{\varepsilon_2}, \Gamma_c}$$

where we recall that $\Psi_N = r \tilde{\Psi}_N$ and $\tilde{\Psi}_N = \phi_0 + a^{3/2} \Psi_{N,0} + a^{3/2} \tilde{E} A + a^2 \tilde{\Psi}_*$, with \tilde{E} and $\tilde{\Psi}_{N,0}$ independent of r and $\tilde{\Psi}_{N,0}$ depends on r only through $B(\lambda A^{3/2})/\lambda$.

Estimates on $(S_{\varepsilon_1}, \Upsilon_{\varepsilon_2})|_{\Gamma=\Gamma_c}$ follow from the above, while

$$\partial_\omega F_{N,\varepsilon_1,\varepsilon_2} = r \partial_\omega \tilde{\Psi}_{N|S_{\varepsilon_1}, \Upsilon_{\varepsilon_2}, \Gamma_c},$$

using that $S_{\varepsilon_1}, \Upsilon_{\varepsilon_2}, \Gamma_c$ are critical points. Set $A_c = \Gamma_c^{2/3}$,

$$\begin{aligned} r^{-1} \partial_\omega F_{N,\varepsilon_1,\varepsilon_2} &= \partial_\omega \psi_0 + a^{3/2} (S_{\varepsilon_1} X \partial_\omega q^{1/3} (1 + \ell) + \Upsilon_{\varepsilon_2} \partial_\omega q^{1/3} + T \partial_\omega (q^{2/3}) A_c) \\ &\quad + a^{3/2} (\partial_\omega E_0 T + \partial_\omega E_1 S_{\varepsilon_1} x + \partial_\omega E_2 \Upsilon_{\varepsilon_2} a) A_c \\ &\quad + a^2 (X S_{\varepsilon_1}^2 f + A_c X f + a^{1/2} X^2 S_{\varepsilon_1} f \\ &\quad \quad + \Upsilon_{\varepsilon_2}^2 f + a^{3/2} \Upsilon_{\varepsilon_2}^2 A_c^2 f + f + a^{1/2} T A_c^2 f). \end{aligned}$$

Differentiating again,

$$\begin{aligned} r^{-1} \partial_\omega^2 F_{N,\varepsilon_1,\varepsilon_2} &= \partial_\omega^2 \psi_0 + a^{3/2} (S_{\varepsilon_1} X \partial_\omega^2 q^{1/3} (1 + \ell) + \Upsilon_{\varepsilon_2} \partial_\omega^2 q^{1/3} + T \partial_\omega^2 (q^{2/3}) A_c) \\ &\quad + a^{3/2} (\partial_\omega S_{\varepsilon_1} X \partial_\omega q^{1/3} (1 + \ell) + \partial_\omega \Upsilon_{\varepsilon_2} \partial_\omega q^{1/3} + T \partial_\omega (q^{2/3}) \partial_\omega A_c) \\ &\quad + a^{3/2} (\partial_\omega^2 E_0 T + \partial_\omega^2 E_1 S_{\varepsilon_1} x + \partial_\omega^2 E_2 \Upsilon_{\varepsilon_2} a) A_c \\ &\quad + a^{3/2} (\partial_\omega E_1 \partial_\omega S_{\varepsilon_1} x + \partial_\omega E_2 \partial_\omega \Upsilon_{\varepsilon_2} a) A_c \\ &\quad + a^{3/2} (\partial_\omega E_0 T + \partial_\omega E_1 S_{\varepsilon_1} x + \partial_\omega E_2 \Upsilon_{\varepsilon_2} a) \partial_\omega A_c \\ &\quad + a^2 (X \partial_\omega (S_{\varepsilon_1}^2 f) + \partial_\omega (A_c f) X + a^{1/2} X^2 \partial_\omega (S_{\varepsilon_1} f) \\ &\quad \quad + \partial_\omega (\Upsilon_{\varepsilon_2}^2 f) + a^{3/2} \partial_\omega (\Upsilon_{\varepsilon_2}^2 A_c f) + \partial_\omega f + T a^{1/2} \partial_\omega (A_c^2 f)). \end{aligned}$$

Therefore we have $r^{-1} \partial_\omega^2 F_{N,\varepsilon_1,\varepsilon_2} = \partial_\omega^2 \psi_0 + \mathfrak{R}$, where the main term is $\partial_\omega^2 \psi_0 = M_0$ with $M_0 = |y| O(1)$ and $M_0^{-1} = |y|^{-1} O(1)$ at the critical points in ω of $F_{N,\varepsilon_1,\varepsilon_2}$, for small $y \neq 0$. the term \mathfrak{R} should be seen as an error term: in fact, one proves easily that for $|y| \geq c_0 t$

and $t \geq c_1 a$, one has

$$|\Re| \leq C\varepsilon_0 t \leq C_1 \varepsilon_0 |y|.$$

Therefore, a stationary phase in ω yields a factor $(h/t)^{(d-2)/2}$.

We now define $\Phi_{N,\varepsilon_1,\varepsilon_2} = \Psi_{N\varepsilon_1,\varepsilon_2|\Gamma_c,\omega_c}$ and $\tilde{\Phi}_{N,\varepsilon_1,\varepsilon_2} = r^{-1}\Phi_{N,\varepsilon_1,\varepsilon_2}$ the phase after stationary phase in ω . As $(S_{\varepsilon_1}, \Upsilon_{\varepsilon_2}, \Gamma_c, \omega_c)$ is a critical point, we have

$$r^2 \partial_r \tilde{\Phi}_{N,\varepsilon_1,\varepsilon_2} = r^2 \partial_r \tilde{\Psi}_{N\varepsilon_1,\varepsilon_2|\Gamma_c,\omega_c}$$

and

$$\lambda \partial_\lambda \tilde{\Psi}_{N\varepsilon_1,\varepsilon_2|\Gamma_c,\omega_c} = a^{3/2} N B_1(\lambda \Gamma_c) \lambda^{-1} \quad \text{with } B_1(U) = -B(U) + U B'(U).$$

Therefore,

$$\begin{aligned} \partial_r^2 \Phi_{N,\varepsilon_1,\varepsilon_2} &= \frac{a^{3/2}}{\lambda} (\lambda \partial_\lambda) (N B_1(\lambda \Gamma_c) \lambda^{-1}) = \frac{a^{3/2}}{\lambda} \frac{N}{\lambda} B_1'(\lambda \Gamma_c) \partial_\lambda (\lambda \Gamma_c) \\ &= \frac{a^{3/2}}{\lambda^2} N B''(\lambda \Gamma_c) \partial_\lambda (\lambda \Gamma_c) \end{aligned}$$

and as $|B''(U)| \geq c_0 U^{-3}$ (as $b_1 \neq 0$ in the asymptotic expansion), we get

$$|\partial_r^2 \Phi_{N,\varepsilon_1,\varepsilon_2}| \geq c_0 a^{3/2} \frac{|N|}{\lambda^2} \Gamma_c^{-1}.$$

Here, exactly as in the previous section, one should perform stationary phase in r only in the regime $\lambda \lesssim |N|$.

we now split into two cases: the first one is T/N bounded, in which case Γ_c is bounded. We then subdivide again: if $|N|/\lambda$ is bounded, then we get

$$|V_{N,1,\varepsilon_1,\varepsilon_2}| \leq \frac{C_0}{h^d} a^{1/2} \left(\frac{a^{3/2}}{h}\right)^{-1/2} T^{-1/2} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \leq \frac{C_0}{h^d} \frac{a^{3/4}}{(\lambda t)^{1/2}} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \leq \frac{C_0}{h^d} \left(\frac{h}{t}\right)^{\frac{d-1}{2}}$$

e.g., the usual dispersion holds true for $V_{N,1,\varepsilon_1,\varepsilon_2}$, and in turn it holds for the sum over N if $\Delta N \leq 2C$ (in particular, if $a > h^{4/7-\varepsilon}$). In the case where $\Delta N \geq 2C$, in fact we may forget about the condition on $|N|/\lambda$ and add the factor from the stationary phase in r , to get

$$|V_{N,1,\varepsilon_1,\varepsilon_2}(t, x, y)| \leq \frac{C_0}{h^d} a^{1/2} \left(\frac{a^{3/2}}{h}\right)^{-1/2} T^{-1/2} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{\lambda^{1/2}}{|N|^{\frac{1}{2}}} \leq \frac{C_0}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^{1/2}}{T}$$

we know that $T \geq \lambda^2$ and $\Delta N \leq 2CT\lambda^{-2}$, so that

$$\left| \sum_{|N| \geq 2} V_{N,1,\varepsilon_1,\varepsilon_2}(t, x, y) \right| \leq \frac{C_0}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^{1/2}}{T} 2C \frac{T}{\lambda^2} \leq \frac{C_0}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} h^{1/3}$$

as $a^{1/2}\lambda^{-2} \leq h^{1/3}$ is nothing but $a > h^{2/3}$.

Now, assume that, for some $N' \in \mathcal{N}_1(x, y, t)$, one has $T'/N' \geq K_0$ for K_0 some large positive constant. Then we established that in that case, we have ((3.9)) $\Delta N \leq CT\lambda^{-2}(T/N)^{-2(3+1/2)}$. If we set $A' = (T'/N')^2$ and $A = (T/N)^2$, one has, if $N \in \mathcal{N}(x, y, t)$,

$$|(A/A') - 1| \leq \varepsilon_0, \quad |N - N'| \leq C + CT'\lambda^{-2}(A')^{-3-1/2},$$

so that $\Delta N \leq C + CT\lambda^{-2}(A')^{-3-1/2}$.

Now, let N be such that $T/N \geq K_0$. In order to perform the stationnary phase in Γ , it is better to change scales and set $\Gamma = \sigma(T/N)^3$, so that we may restrict $0 < c \leq \sigma \leq C$, using $\Gamma_c \sim (T/N)^3$ and discarding an $O(h^\infty)$ contribution from $\sigma > C$. Therefore,

$$V_{N,1,\varepsilon_1,\varepsilon_2} = a^{1/2} \int e^{\frac{i}{h}\Psi_{N,\varepsilon_1,\varepsilon_2}} \mu_{\varepsilon_1,\varepsilon_2,N} d\sigma d\theta + O_{C^\infty}(h^\infty)$$

where the new symbol $\mu_{\varepsilon_1,\varepsilon_2,N}$ is such that, for all k and κ ,

$$|\partial_\theta^\kappa \partial_\sigma^k \mu_{\varepsilon_1,\varepsilon_2,N}| \leq C_{\kappa,k} \frac{T}{N} \sigma^{-2/3-k}.$$

As we have

$$|\partial_\sigma^2 \Psi_{N,\varepsilon_1,\varepsilon_2}| \geq c_0 \left(\frac{T}{N}\right)^2 T a^{3/2},$$

we now get by stationnary phase

$$|V_{N,1,\varepsilon_1,\varepsilon_2}| \leq \frac{C_0}{h^d} a^{1/2} \inf(1, (\frac{|N|}{\lambda \Gamma_c})^{-1/2}) \frac{T}{N} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{N}{T} \frac{1}{(\lambda T)^{1/2}}$$

and one remarks that $\lambda \Gamma_c N^{-1} \sim \lambda T^3 N^{-4}$; writing $T/N^2 = (T/N)^2/T$ and choosing A' , N' as above, we get

$$|V_{N,1,\varepsilon_1,\varepsilon_2}| \leq \frac{C_0}{h^d} a^{1/2} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{A'}{T}$$

Notice how our estimate of each $V_{N,1,\varepsilon_1,\varepsilon_2}$ is worst than before, but we do have a better estimate on $|\mathcal{N}_1(x, y, t)|$ to compensate for it.

We then sum over $N \in \mathcal{N}_1$: either $\Delta N \geq 2C$, and from $\Delta N \leq 2CT\lambda^{-2}(A')^{-3-1/2}$, we get

$$\left| \sum_{|N| \geq 2} V_{N,1,\varepsilon_1,\varepsilon_2}(t, x, y) \right| \leq \frac{C_0}{h^d} a^{1/2} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{A'}{T} T \lambda^{-2} (A')^{-3-1/2}$$

As A' is large and $a^{1/2} \lambda^{-2} \leq h^{1/3}$, we are done.

If on the other hand $\Delta N \leq 2C$, we directly get

$$\left| \sum_{|N| \geq 2} V_{N,1,\varepsilon_1,\varepsilon_2}(t, x, y) \right| \leq \frac{C_0}{h^d} a^{1/2} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{1}{T^{1/2} \lambda^{1/2}}$$

and from $T \geq T_0 > 0$ and $a^{1/2}\lambda^{-1/2} \leq h^{1/3}$ we are done. \square

We now proceed with the special case $|N| = 1$ and only highlight the differences with the previous case $|N| \geq 2$. Recall that here we have $A > A_1$ where A_1 is a (large) constant. In order for the phase $\Psi_{1,\varepsilon_1,\varepsilon_2}$ to have a critical point at $A = A_c$, T will have to be small, actually $T \sim A_c^{-1/2}$ and one may compute $(\partial_A^2 \Psi_{1,\varepsilon_1,\varepsilon_2})(A_c) \sim T^3$. Changing variable $A = T^{-2}\sigma$, one easily checks that the rescaled symbols in the definition (3.33) of $V_{1,1}$ are classical symbols of degree zero in σ , uniformly in T , provided $\sigma > \sigma_0 > 0$. By non degenerate stationary phase, we get

$$(3.34) \quad |V_{1,1}| \leq \frac{C}{h^d} \left(\frac{h}{t} \right)^{\frac{d-2}{2}} \frac{1}{\lambda^{1/2} T^{1/2}}.$$

Again, we have $(\lambda T)^{-1/2} \lesssim h^{1/3}$ and we are done. \square

Remark 3.13. One should notice that $h^{1/3} \leq a^{1/8} h^{1/4}$ as $a > h^{2/3}$. As such, at least for $|t| \leq C\sqrt{a}$, (3.34) is at least as good as the bound where $h^{1/3}$ gets replaced by $a^{1/4}(h/t)^{1/4}$, as stated for V_1 in [8].

4. CONSTRUCTION OF WHISPERING GALLERY MODES

We recall the formula of the parametrix $\mathcal{P}_{h,a}$ obtained by the distributional bracket

$$(4.1) \quad \mathcal{P}_{h,a}(t, x, y) = \left\langle \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, K_\omega(g_{h,a})(t, x, y) \right\rangle_\omega$$

and the Airy-Poisson formula (2.14) which yields

$$(4.2) \quad \mathcal{P}_{h,a}(t, x, y) = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} K_{\omega_k}(g_{h,a})(t, x, y).$$

We introduce (2.19) (the formula of $g_{h,a}$ before the stationary phase with respect to s, α) in the integral defining K_{ω_k} : the integral with respect to y' yields a factor h^{d-1} (and $\theta' = \theta$), while applying the stationary phase with respect to (α, ϱ) provides a factor h since the critical points $(\alpha = h^{2/3}\omega_k, \varrho = \partial_\alpha \Phi(a, 0, \theta, h^{2/3}\omega_k, \sigma))$ are non-degenerate. We obtain

$$(4.3) \quad K_{\omega_k}(g_{h,a})(t, x, y) = \frac{h^{-d-1/3}}{2\pi} \int e^{\frac{i}{h}(t\rho(h^{2/3}\omega_k, \theta) + \Phi(x, y, \theta, h^{2/3}\omega_k, s) - \Phi(a, 0, \theta, h^{2/3}\omega_k, s'))} p_{0,h} ds ds' d\theta,$$

where Φ has been defined in (2.11) and where the symbol is of the form

$$p_{0,h} = p_{0,h}(x, a, y, \theta/h, \omega_k, s/h^{1/3}, s'/h^{1/3}, h),$$

with $p_{0,h}(x, a, y, \eta, \omega_k, \sigma, \sigma', h)$ of order 0. Using (4.2) and (4.3), there exists a symbol $\check{p}_{0,h}$ of order 0 such that the parametrix takes the following form

$$(4.4) \quad \mathcal{P}_{h,a}(t, x, y) = \sum_{k \in \mathbb{N}^*} \frac{h^{-d+1/3}}{L'(\omega_k)} \int e^{\frac{i}{h}t\rho(h^{2/3}\omega_k, \theta)} \check{p}_{0,h} G(x, y, \theta/h, \omega_k) \overline{G(a, 0, \theta/h, \omega_k)} d\theta$$

where G is the one introduced in (2.3) (there is a factor $h^{-\frac{1}{3}}$ inside each G - which appears from the change of variables with respect to s , (s' , respectively) therefore the exponent of h had changed into $(-d + 1/3)$).

4.1. Pseudo-differential calculus; construction of gallery modes.

Definition 4.1. Let G be defined in (2.3), satisfying (2.4) and let like before $\tau = \tau(\omega, \eta) = \sqrt{|\eta|^2 + \omega q(\eta)^{2/3}}$ and $\rho(\alpha, \theta) = \sqrt{|\theta|^2 + \alpha q(\theta)^{2/3}}$, where, for $\eta = \theta/h$ and $\alpha = h^{2/3}\omega$ we have $\tau(\omega, \eta) = \rho(\alpha, \theta)/h$. We set

$$(4.5) \quad e(x, y, \eta, \omega) = \frac{1}{2\pi\sqrt{L'(\omega)}} \int q(\eta)^{1/6} e^{i(\sigma^3/3 + \sigma(xq^{1/3}(\eta) - \omega) + \rho\Gamma(x, y, \sigma q^{1/3}(\eta)/\rho, \eta/\rho))} p d\sigma.$$

Then $e(x, y, \eta, \omega) = \frac{q(\eta)^{1/6}}{\sqrt{L'(\omega_k)}} e^{-iy\eta} G(x, y, \eta, \omega)$ from (2.9); we now define, for $g \in L^2(\mathbb{R}^{d-1})$,

$$F_k(g)(x, y) := \frac{1}{(2\pi)^{d-1}} \int e^{i(y-z)\eta} e(x, y, \eta, \omega_k) g(z) d\eta dz = \int G(x, y, \eta, \omega_k) \hat{g}(\eta) d\eta.$$

For a given N and $1 \leq k \leq N$ we consider the operator

$$j_N : (L^2(\mathbb{R}^{d-1}))^N \rightarrow L^2(\mathbb{R}^{d-1}(L^2(\mathbb{R}_+)))$$

defined as follows

$$j_N(g_1, \dots, g_N) = \sum_{k=1}^N F_k(g_k).$$

Proposition 4.2. *For $N \lesssim h^{-\theta}$, $\theta < \frac{1}{10}$, the following holds*

$$(4.6) \quad \|j_N(g_1, \dots, g_N)\|_{L^2(\Omega)} \simeq \|(g_1, \dots, g_N)\|_{(L^2(\mathbb{R}^{d-1}))^N}.$$

Before the proof of the proposition we shall slightly modify j_N by replacing it by $j_N \circ \mathfrak{L}$, where \mathfrak{L} is an isometry on L^2 suitably chosen. For $g \in L^2(\mathbb{R}^{d-1})$ set

$$\mathfrak{L}(g)(z) := \int e^{i(z-y')\eta - i|\eta|B_0(y', \eta/|\eta|)} g(y') dy' d\eta,$$

where B_0 is the first term in the development of B in (5.2) near the glancing region \mathcal{G} , homogeneous of degree 0, such that the phase function $\Gamma = B_0 + (|\eta| - 1)B_2 + xA_1 + O(\dots)$.

Lemma 4.3. *\mathfrak{L} is an isometry on $L^2(\mathbb{R}^{d-1})$.*

The lemma follows using the form of B_0 from Lemma 5.2 and (5.7).

Remark 4.4. Our goal here is to get rid of the term homogeneous of degree zero of the phase function of $e(x, y, \eta, \omega_k)$. In fact, in order to prove Proposition 4.2, we shall be reduced to estimate the derivatives of $e(\cdot, \omega_k)$ with respect to (y, η) : while in the Friedlander case there is no dependence on y in the formula of e_k , in the general situation the derivative

with respect to y yields a factor which is very bad for our estimates and which comes from the homogeneous of degree zero part of the phase $\tau\Gamma(x, y, \sigma q^{1/3}(\eta)/\tau, \eta/\tau)$. We have:

$$\begin{aligned}\tau\Gamma(x, y, \sigma q^{1/3}(\eta)/\tau, \eta/\tau) &= \tau(\omega, \eta)B_0(y, \eta/|\eta|) + \tau(|\eta|/\tau - 1)B_2 + \dots \\ &= \tau(\omega, \eta)B_0(y, \eta/|\eta|) + \frac{\omega}{2}|\eta|^{1/3}B_2 + \dots\end{aligned}$$

Remark 4.5. Since \mathfrak{L} is an isometry, it will be enough to prove Proposition 4.2 where j_N is replaced by \tilde{j}_N , where we have set

$$\tilde{j}_N(g_1, \dots, g_N) := j_N(\mathfrak{L}(g_1), \dots, \mathfrak{L}(g_N)) = \sum_{k=1}^N F_k \circ \mathfrak{L}(g_k).$$

We set $\tilde{e}_k(x, y, \eta) = e^{-i|\eta|B_0(y, \eta/|\eta|)}e(x, y, \eta, \omega_k)$ and we compute

$$\begin{aligned}F_k \circ \mathfrak{L}(g)(x, y) &= \frac{1}{(2\pi)^{d-1}} \int_z \int_\eta e^{i(y-z)\eta} e(x, y, \eta, \omega_k) \mathfrak{L}(g)(z) d\eta dz \\ &= \frac{1}{(2\pi)^{d-1}} \int_{y'} \int_\eta e^{i(y-y')\eta - i|\eta|B_0(y', \eta/|\eta|)} e(x, y, \eta, \omega_k) g(y') d\eta dy' \\ &= \frac{1}{(2\pi)^{d-1}} \int_{y'} \int_\eta e^{i(y-y')\eta + i|\eta|(B_0(y, \eta/|\eta|) - B_0(y', \eta/|\eta|))} \tilde{e}_k(x, y, \eta) g(y') d\eta dy' \\ &= \frac{1}{(2\pi)^{d-1}} \int_{y'} \int_\eta e^{i(y-y')\Theta} \tilde{e}_k(x, y, \Theta) \left| \frac{d\Theta}{d\eta} \right| g(y') d\Theta dy',\end{aligned}$$

where in the last line we have set $(y - y')\eta + |\eta|(B_0(y, \eta/|\eta|) - B_0(y', \eta/|\eta|)) = (y - y')\Theta(y, y', \eta)$ whose Jacobian satisfies $\left| \frac{d\Theta}{d\eta} \right| = 1 + O(|\nabla_y B_0|)$.

Proof. (of Proposition 4.2) In order to show that for some $\theta \in [0, 1]$ and $N \leq h^{-\theta}$ we do have (4.6) it will be enough to prove that $\tilde{j}_N^* \tilde{j}_N$ is bounded and invertible on $(L^2(\mathbb{R}^{d-1}))^N$. In the following we compute this operator explicitly. Let $f \in L^2(\Omega)$, then for $(g_1, \dots, g_N) \in L^2(\mathbb{R}^{d-1})$ we have

$$\begin{aligned}\langle (g_1, \dots, g_N), \tilde{j}_N^*(f) \rangle_{(L^2(\mathbb{R}^{d-1}))^N} &= \langle \tilde{j}_N(g_1, \dots, g_N), f \rangle_{L^2(\mathbb{R}^{d-1})(L^2(\mathbb{R}_+))} \\ &= \int_\Omega \sum_{k=1}^N F_k(\mathfrak{L}(g_k))(x, y) \overline{f(x, y)} dx dy \\ &= \sum_{k=1}^N \int_{\mathbb{R}^{d-1}} g_k(y) \overline{(F_k \circ \mathfrak{L})^*(f)}(y) dy.\end{aligned}$$

Setting $\tilde{F}_k = F_k \circ \mathfrak{L}$, we obtain $\tilde{j}_N(f) = (\overline{\tilde{F}_1^*(f)}, \dots, \overline{\tilde{F}_N^*(f)})$, hence

$$\tilde{j}_N^*(\tilde{j}_N(g_1, \dots, g_N)) = \left(\tilde{F}_1^* \left(\sum \tilde{F}_k(g_k) \right), \dots, \tilde{F}_N^* \left(\sum \tilde{F}_k(g_k) \right) \right).$$

The matrix M_N associated to the operator $\tilde{j}_N^* \circ \tilde{j}_N : (L^2(\mathbb{R}^{d-1}))^N \rightarrow (L^2(\mathbb{R}^{d-1}))^N$ is given by $M_N := (\tilde{F}_j^* \circ \tilde{F}_k)_{j,k \in \{1, \dots, N\}}$. We compute the operators $\tilde{F}_j^* \tilde{F}_k$ explicitly. Our goal is

to show that M_N is a matrix of pseudo-differential operators in y , bounded and invertible, uniformly with respect to $N \lesssim h^{-\theta}$ if $\theta < \frac{1}{10}$. For $g \in L^2(\mathbb{R}^{d-1})$ and $f \in L^2(\Omega)$ we have $\langle \tilde{F}_k(g), f \rangle_{L^2(\Omega)} = \langle g, \tilde{F}_k^*(f) \rangle_{L^2(\mathbb{R}^{d-1})}$, therefore

$$\begin{aligned} \langle \tilde{F}_k(g), f \rangle_{L^2(\Omega)} &= \int_{\Omega} \left(\frac{1}{(2\pi)^{d-1}} \int_z \int_{\eta} e^{i(y-z)\eta} e(x, y, \eta, \omega_k) \mathfrak{L}(g)(z) d\eta dz \right) \overline{f(x, y)} dx dy \\ &= \int_{\Omega} \left(\frac{1}{(2\pi)^{d-1}} \int_{y'} \int_{\eta} e^{i(y-y')\eta - i|\eta|B_0(y', \eta/|\eta|)} e(x, y, \eta, \omega_k) g(y') d\eta dy' \right) \overline{f(x, y)} dx dy \\ &= \int_{y'} g(y') \left(\frac{1}{(2\pi)^{d-1}} \int_{\Omega} \int_{\eta} e^{i(y'-y)\eta + i|\eta|B_0(y', \eta/|\eta|)} \overline{e(x, y, \eta, \omega_k)} f(x, y) d\eta dx dy \right) dy' \\ &= \int_z g(z) \overline{\tilde{F}_k^*(f)(z)} dz = \langle g, \tilde{F}_k^*(f) \rangle_{L^2(\mathbb{R}^{d-1})}, \end{aligned}$$

which yields the formula of $\tilde{F}_k^*(f)$. We compute $\tilde{F}_j^* \circ \tilde{F}_k(g)(z)$ for $g \in L^2(\mathbb{R}^{d-1})$, $z \in \mathbb{R}^{d-1}$:

$$\begin{aligned} \tilde{F}_j^* \circ \tilde{F}_k(g)(z) &= \frac{1}{(2\pi)^{d-1}} \int_{\Omega} \int_{\eta} e^{i(z-y)\eta + i|\eta|B_0(z, \eta/|\eta|)} \overline{e(x, y, \eta, \omega_j)} d\eta \tilde{F}_k(g)(x, y) dx dy \\ (4.7) \quad &= \int_{z'} M_{j,k}(z, z') g(z') dz', \end{aligned}$$

where we have

$$\begin{aligned} M_{j,k}(z, z') &= \frac{1}{(2\pi)^{d-1}} \int_{y, \eta, \eta'} e^{i \left[(z-y)\eta + |\eta|(B_0(z, \eta/|\eta|) - B_0(y, \eta/|\eta|)) \right] - i \left[(z'-y)\eta' + |\eta'|(B_0(z', \eta'/|\eta'|) - B_0(y, \eta'/|\eta'|)) \right]} \\ &\quad \times \int_0^\infty \overline{\tilde{e}_j(x, y, \eta)} \tilde{e}_k(x, y, \eta') dx d\eta' d\eta dy. \end{aligned}$$

Let like before $\Theta = \Theta(z, y, \eta)$ (respectively $\Theta' = \Theta'(z', y, \eta')$) such that

$$(z - y)\Theta = (z - y)\eta + |\eta|(B_0(z, \eta/|\eta|) - B_0(y, \eta/|\eta|)),$$

then $|\nabla_\eta \Theta| \simeq 1 + O(|\nabla_y B_0|)$ is close to 1 and hence $|\nabla_{\Theta} \eta| \simeq 1 + O(|\nabla_y B_0|)$ also, hence

$$(4.8) \quad M_{j,k}(z, z') = \frac{1}{(2\pi)^{d-1}} \int_{y, \Theta, \Theta'} e^{i(z-y)\Theta - i(z'-y)\Theta'} |\nabla_{\Theta} \eta| |\nabla_{\Theta'} \eta'| \int_0^\infty \overline{\tilde{e}_j(x, y, \Theta)} \tilde{e}_k(x, y, \Theta') dx d\Theta' d\Theta dy.$$

Remark 4.6. Notice that in the case of the Friedlander operator we have $B_0 = 0$ everywhere and we do not need to introduce the operator \mathfrak{L} (which is the identity in this case); moreover the corresponding gallery modes e_j, e_k being orthogonal, the integral over the normal variable x equals $\delta_{j,k}$ and hence $F_j^* \circ F_k = 0$ for $j \neq k$, while $F_k^* \circ F_k(g) = g$. The corresponding matrix satisfies $M_N = I_N$.

Taking in (4.8) $y = z' + w$, $\Theta' = \Theta - \zeta$ yields

$$(4.9) \quad M_{j,k}(z, z') = \int_{\Theta} e^{i(z-z')\Theta} m_{j,k}(z, z', \Theta) d\Theta,$$

$$m_{j,k}(z, z', \Theta) = \frac{1}{(2\pi)^{d-1}} \int_{\zeta} \int_w e^{-i w \zeta} \sigma \int_0^\infty \overline{\tilde{e}_j(x, z' + w, \Theta)} \tilde{e}_k(x, z' + w, \Theta - \zeta) dx dw d\zeta,$$

where we set

$$(4.10) \quad \sigma = \sigma(z, z', \Theta; w, \zeta) = |\nabla_{\Theta} \eta|(z, z' + w, \Theta) |\nabla_{\Theta'} \eta'| (z', z' + w, \Theta - \zeta).$$

Notice that in (4.9) we can replace $m_{j,k}(z, z', \Theta)$ by $m_{j,k}(z', z', \Theta)$ without changing the integral modulo $O(\Theta^{-\infty})$. Since $\Theta = \frac{\theta}{h}$, for some $\theta \simeq 1$, $h \in (0, 1]$, taking $\zeta = \varrho/h$, $\tilde{\sigma}(z', \cdot) = \sigma(z', z', \cdot)$, we replace $m_{j,k}(z', z', \Theta)$ by $\tilde{m}_{j,k}(z', \theta, h)$ given by

$$\tilde{m}_{j,k}(z', \theta, h) = \frac{1}{(2\pi h)^{d-1}} \int_{\varrho} \int_w e^{-\frac{i}{h} w \varrho} \tilde{\sigma} \int_0^\infty \overline{\tilde{e}_j(x, z' + w, \theta/h)} \tilde{e}_k(x, z' + w, (\theta - \varrho)/h) dx dw d\varrho.$$

Proposition 4.7. *For $N \leq h^{-\theta}$ for $\theta \in [0, \frac{1}{10})$, the matrix M_N can be written under the form $M_N = D_N + R_N$, where*

- (1) $D_N = \text{diag}(\tilde{F}_k^* \circ \tilde{F}_k)_{k \in \{1, \dots, N\}}$ is a matrix of pseudo-differential operators;
- (2) R_N satisfies $\|D_N^{-1} R_N\| \leq \frac{1}{2}$.

Remark 4.8. Notice that it will be easier to estimate the Hilbert-Schmidt norm of the elements of M_N instead of their L^2 norm. Recall that if $k : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ is such that $\int_{\mathbb{R}^{2n}} |k(x, y)|^2 dx dy < \infty$, then the operator $K : L^2(\mathbb{R}^n, \mathbb{C}) \rightarrow L^2(\mathbb{R}^n, \mathbb{C})$ by

$$Kg(z) = \int k(z, z') g(z') dz'$$

is Hilbert-Schmidt, with Hilbert-Schmidt norm $\|K\|_{HS} = \|k\|_{L^2} \geq \|K\|_{L^2 \rightarrow L^2}$. From the formula (4.7), it follows that in order to estimate the Hilbert-Schmidt norms of $\tilde{F}_j^* \circ \tilde{F}_k$ it is enough to estimate the L^2 norms of their kernels $M_{j,k}$ defined in (4.9). Since $\|M_{j,k}\|_{L^2_{z, z'}} = \|\tilde{m}_{j,k}(z', \theta, h)\|_{L^2_{z', \theta}}$, we estimate the last norms first for $j = k$, then for $j \neq k$.

Definition 4.9. We recall the definition of the class of symbols \mathcal{S}_γ :

$$\mathcal{S}_\gamma := \{a \in C^\infty \mid |\partial^\beta a(w, \varrho, h)| \leq C_\beta h^{-\gamma|\beta|}\}.$$

Proposition 4.10. *Let $a(w, \varrho, h) : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ be smooth, $a \in \mathcal{S}_{\frac{1}{2}-\epsilon}$, for some small $\epsilon > 0$.*

Then we have

$$\frac{1}{(2\pi h)^n} \int_{\varrho} \int_w e^{-\frac{i}{h} w \varrho} a(w, \varrho, h) dw d\varrho \simeq \sum \frac{h^{|\beta|}}{|\beta|!} \left(\frac{D_w, D_\varrho}{i} \right)^{|\beta|} a(w, \varrho, h) \Big|_{(w, \varrho) = (0, 0)}.$$

We define

$$(4.11) \quad a_{j,k}(z', \theta; w, \varrho; h) := \tilde{\sigma}(z', \theta/h; w, \varrho/h) \int_0^\infty \overline{\tilde{e}_j(x, z' + w, \theta/h)} \tilde{e}_k(x, z' + w, (\theta - \varrho)/h) dx.$$

Using Proposition 4.10, we intend to write $\tilde{m}_{j,k}(z', \theta, h)$ under the form

$$(4.12) \quad \tilde{m}_{j,k}(z', \theta, h) \simeq \sum_{\beta} \frac{h^{|\beta|}}{i^{|\beta|} |\beta|!} \partial_w^\beta \partial_\varrho^\beta a_{j,k}(z', \theta; w, \varrho; h) \Big|_{w=0, \varrho=0}.$$

We now prove Proposition 4.7), starting with proving that D_N is a matrix of pseudo-differential operators. For, we have to show that $\exists C > 0$ independent on $N \lesssim h^{-\theta}$, such that the following holds :

$$\|\tilde{F}_k^* \circ \tilde{F}_k\|_{HS} \leq C \text{ and } \|(\tilde{F}_k^* \circ \tilde{F}_k)^{-1}\|_{HS} \leq C.$$

Notice that this is equivalent to saying that $\tilde{F}_k^* \circ \tilde{F}_k : L^2(\mathbb{R}^{d-1}) \rightarrow L^2(\mathbb{R}^{d-1})$ is a pseudo-differential operator, uniformly elliptic w.r.t. $1 \leq k \leq N \lesssim h^{-\theta}$. From (4.7), (4.9) and Remark 4.8 it follows that $\|\tilde{F}_k^* \circ \tilde{F}_k\|_{HS} = \|M_{k,k}\|_{L^2_{z',\theta}} = \|\tilde{m}_{k,k}(z', \theta, h)\|_{L^2_{z',\theta}}$ and using (4.12) (for values $k \leq N \lesssim h^{-\theta}$ for which we are allowed to apply Proposition 4.10, that we shall determine below) we get

$$\begin{aligned} \|a_{k,k}(z', \theta; 0, 0; h)\|_{L^2_{z',\theta}} - \left\| \sum_{|\beta| \geq 1} \frac{h^{|\beta|}}{|\beta|!} \partial_w^\beta \partial_\varrho^\beta a_{k,k}(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right\|_{L^2_{z',\theta}} &\leq \|\tilde{m}_{k,k}(z', \theta, h)\|_{L^2_{z',\theta}} \\ &\leq \|a_{k,k}(z', \theta; 0, 0; h)\|_{L^2_{z',\theta}} + \left\| \sum_{|\beta| \geq 1} \frac{h^{|\beta|}}{|\beta|!} \partial_w^\beta \partial_\varrho^\beta a_{k,k}(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right\|_{L^2_{z',\theta}}. \end{aligned}$$

We start with the L^2 norm of $a_{k,k}(z', \theta; 0, 0; h)$, which equals

$$a_{k,k}(z', \theta; 0, 0; h) = \tilde{\sigma}(z', z', \theta/h; 0, 0) \|\tilde{e}_k(\cdot, z', \theta/h)\|_{L_x^2}^2.$$

Since e_k are L^2 normalised in the normal variable, the same holds for \tilde{e}_k , as the phase function B_0 does not depend on x . On the other hand, $\tilde{\sigma} = \sigma|_{z=z'}$ is an elliptic symbol as defined in (4.10). Therefore $a_{k,k}$ is elliptic and $\|a_{k,k}(z', \theta; 0, 0; h)\|_{L^2_{z',\theta}} \simeq 1$. In order Proposition 4.10 to apply, it remains to check that for $|\beta| \geq 1$ we have

$$h^{|\beta|} \left\| \partial_w^\beta \partial_\varrho^\beta a_{k,k}(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right\|_{L^2_{z',\theta}} \leq h^\epsilon.$$

for some $\epsilon > 0$ as small as we want. We need the following lemma:

Lemma 4.11. *Let $\rho = (y, \theta)$, then the following holds, uniformly for $h^{2/3}\omega_k \ll 1$:*

$$(4.13) \quad \|\partial_\rho^\beta \tilde{e}_k(\cdot, y, \theta/h)\|_{L^2(x \geq 0)} \lesssim \omega_k^{\frac{3|\beta|}{2}} + \left(\omega_k/h^{1/3}\right)^{|\beta|}.$$

Using (4.11) for $j = k$, the properties of $\tilde{\sigma}$ and (4.13), we get

$$h^{|\beta|} \left\| \partial_w^\beta \partial_\varrho^\beta a_{k,k}(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right\|_{L^2_{z',\theta}} \lesssim C_\beta h^{\frac{|\beta|}{3}} k^{4|\beta|/3}, \quad \forall |\beta| \geq 1,$$

and if we ask this to be smaller than a small, positive power of h for all $k \lesssim h^{-\theta}$ it follows that we must have $\theta < \frac{1}{4}$. It is clear that the proof will follow the same path for the inverse operator $(\tilde{F}_k^* \circ \tilde{F}_k)^{-1}$, with the same condition on θ .

For the second part of the proof it will be enough to show that $\|R_N\|_{(L^2(\mathbb{R}^{d-1}))^N} \leq h^\epsilon$ for some small $\epsilon > 0$. Let $j \neq k$ and write

$$\|\tilde{m}_{j,k}(z', \theta, h)\|_{L^2_{z',\theta}} \leq \|a_{j,k}(z', \theta; 0, 0; h)\|_{L^2_{z',\theta}} + \left\| \sum_{|\beta| \geq 1} \frac{h^{|\beta|}}{|\beta|!} \partial_w^\beta \partial_\varrho^\beta a_{j,k}(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right\|_{L^2_{z',\theta}}.$$

To estimate $\|a_{j,k}\|_{L^2_{z',\theta}}$ we use the Cauchy-Schwartz estimate and the following lemma :

Lemma 4.12. *For $j \neq k$ such that $h^{2/3}\omega_k \ll 1$ and $h^{2/3}\omega_j \ll 1$, the following holds:*

$$(4.14) \quad \langle \tilde{e}_k, \tilde{e}_j \rangle_{L^2(x>0)} \lesssim h^{1/3} \left(1 + \frac{1}{|\omega_k - \omega_j|} \right).$$

Since z' remains bounded, θ is close to one and $\tilde{\sigma}$ is a smooth symbol, this gives

$$\|a_{j,k}\|_{L^2_{z',\theta}} \lesssim \langle \tilde{e}_k, \tilde{e}_j \rangle_{L^2(x>0)} \lesssim h^{1/3} \left(1 + \frac{1}{|\omega_k - \omega_j|} \right),$$

$$\sum_{j \neq k, j, k \leq N} \|a_{j,k}\|_{L^2_{z',\theta}} \lesssim 2h^{1/3} \sum_{j < k \leq N} \left(1 + \frac{1}{|\omega_k - \omega_j|} \right) \lesssim h^{1/3} N^2.$$

Taking the derivatives w.r.t. (w, ϱ) and using (4.13) and (4.14) yields:

$$(4.15) \quad \partial_w^\beta \partial_\varrho^\beta a_{j,k}(z', \theta, ; w, \varrho; h)|_{(w,\varrho)=(0,0)} = \sum_{l_1+l_2=0}^{|\beta|} \sum_{q=0}^{|\beta|} C_{|\beta|}^q \partial_{(w,\varrho)}^{(|\beta|-l_1-l_2, |\beta|-q)} \tilde{\sigma}(z', \theta/h; w, \varrho/h) \\ \times \int_0^\infty \partial_w^{l_1} \tilde{e}_j(x, z' + w, \theta/h) \overline{\partial_{(w,\varrho)}^{(l_2,q)} \tilde{e}_k(x, z' + w, (\theta - \varrho)/h)} dx \Big|_{(w,\varrho)=(0,0)}.$$

To estimate (4.15) we now apply Cauchy Schwartz and use (4.13) which give bounds for the integral over x (and we also use the properties of B_0 to deduce that the symbol $\tilde{\sigma}$ is bounded together with all its derivatives). For $1 \leq j, k \leq N$ such that $h^{2/3}\omega_j \ll 1$ and $h^{2/3}\omega_k \ll 1$ we write

$$\|R_N\|_{HS} \leq \sum_{j,k \leq N, j \neq k} \|\tilde{m}_{j,k}(z', \theta, h)\|_{L^2_{z',\theta}} \leq h^{1/3} N^2 + \sum_{|\beta| \geq 1} h^{|\beta|} \left\| \sum_{j \neq k, j, k \leq N} \partial_w^\beta \partial_\varrho^\beta a_{j,k}(z', \theta, ; h)|_{(0,0)} \right\|_{L^2} \\ \leq h^{1/3} N^2 + \sum_{|\beta| \geq 1} C_\beta h^{|\beta|} \sum_{j \neq k, j, k \leq N} \left(\frac{(jk)^{2|\beta|/3}}{h^{2|\beta|/3}} + \frac{k^{4|\beta|/3}}{h^{2|\beta|/3}} \right) \\ \leq h^{1/3} N^2 + \sum_{|\beta| \geq 1} h^{\frac{|\beta|}{3}} \left(\sum_{1 \leq j \leq N} j^{2/3} \right)^2 \simeq h^{1/3} N^2 + \sum_{|\beta| \geq 1} h^{\frac{|\beta|}{3}} N^{2+4|\beta|/3}.$$

If we impose the Hilbert-Schmidt norm of R_N to satisfy $\|R_N\|_{HS} \leq h^\epsilon$ for some small $\epsilon > 0$, we obtain a restriction on θ , where $N \leq h^{-\theta}$: we must have

$$\frac{|\beta|}{3} - \theta(2 + \frac{4|\beta|}{3}) > 0 \quad \forall |\beta| \geq 1,$$

which gives $\theta < \frac{|\beta|}{6+4|\beta|}$ and for $|\beta| = 1$ this gives $\theta < \frac{1}{10}$. For values of $N \ll h^{-1/10}$ we also have $h^{1/3}N^2 \ll h^{\frac{1}{3}-\frac{1}{5}} = h^{\frac{2}{15}}$ and therefore this term is also very small. Since $\|R_N\|_{HS} \lesssim h^\epsilon$ for $\frac{1}{3} - \theta(2 + \frac{4}{3}) \geq \epsilon > 0$, the Hilbert-Schmidt norm of $D_N^{-1} \circ R_N$ will be also as small. \square

4.2. Dispersion estimates for a small. Until now we have provided sharp uniform dispersive bounds for the Green function $\mathcal{P}_{h,a}$ for all $h^{2/3} \ll a \leq 1$, using formula (4.1) (where the sum is indexed over the number of reflections on the boundary). In this part we provide dispersive estimates for $\mathcal{P}_{h,a}$ for a very small, $a \in (0, h^{2/3(1-\epsilon)}]$ for some $\epsilon > 0$; in order to do this, we use the formula (4.2) (in terms of gallery modes, constructed above).

From formula (4.4) and the definition (4.5) (where the factor $q(\eta)^{1/6}$ gives a factor $h^{-1/3}$, which appears twice) it follows that

(4.16)

$$\begin{aligned} \mathcal{P}_{h,a}(t, x, y) &= \sum_{k \in \mathbb{N}^*} h^{-(d-1)} \int e^{\frac{i}{h}(y\theta + t\rho(h^{2/3}\omega_k, \theta))} \check{p}_{0,h} e(x, y, \theta/h, \omega_k) \overline{e(a, 0, \theta/h, \omega_k)} d\theta \\ &= \sum_{k \in \mathbb{N}^*} h^{-(d-1)} \int e^{\frac{i}{h}(y\theta + |\theta|B_0(y, \theta/|\theta|) + t\rho(h^{2/3}\omega_k, \theta))} \check{p}_{0,h} \tilde{e}(x, y, \theta/h, \omega_k) \overline{\tilde{e}(a, 0, \theta/h, \omega_k)} d\theta \end{aligned}$$

since $e(a, 0, \theta/h, \omega) = \frac{e^{-i|\eta|B_0(0, \eta/|\eta|)}}{\sqrt{L'(\omega_k)}} q(\eta)^{1/6} G(a, 0, \eta, \omega) = \tilde{e}(a, 0, \theta/h, \omega)$ as $B_0(0, \eta/|\eta|) = 0$.

Remark 4.13. Notice that formula (4.16) coincides with the parametrix $\mathcal{P}_{F,h,a}$ if Δ is the Friedlander model operator $\Delta_F = \partial_{xx}^2 + (1+x)\Delta_y$: indeed, in this case the eigenfunctions $\{e_k(x, \eta)\}_{k \geq 1}$ of the operator $-\partial_{xx}^2 + (1+x)|\eta|^2$ associated to the eigenvalues $\lambda_k(\eta) = |\eta|^2 + \omega_k|\eta|^{4/3}$ are given by

$$e_k(x, \eta) = \frac{f_k}{k^{1/6}} |\eta|^{1/3} Ai(|\eta|^{2/3}x - \omega_k),$$

and setting $\eta = \theta/h$ yields

$$\mathcal{P}_{F,h,a}(t, x, y) = \sum_{k \in \mathbb{N}^*} \frac{1}{h^{d-1}} \int_{\mathbb{R}^{d-1}} e^{it\sqrt{\lambda_k(\theta/h)}} e^{iy\theta/h} \psi(\theta) e_k(x, \theta/h) \overline{e_k(a, \theta/h)} d\theta.$$

To estimate $\mathcal{P}_{h,a}$ for $a \in (0, h^{2/3(1-\epsilon)}]$, $\epsilon > 0$ small, we distinguish the three cases. The main contribution corresponds to values $k \lesssim \lambda = \frac{a^{3/2}}{h} \lesssim h^{-\epsilon}$ and we need to deal separately with two situations, according to whether t is small or not.

- When the time is very small, $t \lesssim h^\epsilon$, the stationary phase does not apply: in this case we use Lemma [8, Lemma 3.5] and its the corresponding version involving the derivatives of the Airy function, that we summarise here:

Lemma 4.14. *There exists C_0 such that for $L \geq 1$ the following holds true*

$$(4.17) \quad \sup_{b \in \mathbb{R}} \left(\sum_{1 \leq k \leq L} k^{-1/3} Ai^2(b - \omega_k) \right) \leq C_0 L^{1/3}.$$

Moreover, we have

$$(4.18) \quad \sup_{b \in \mathbb{R}_+} \left(\sum_{1 \leq k \leq L} k^{-1/3} h^{2/3} A i'^2 (b - \omega_k) \right) \leq C_0 h^{2/3} L,$$

$$(4.19) \quad \sup_{b \in \mathbb{R}_-} \left(\sum_{1 \leq k \leq L} k^{-1/3} h^{2/3} A i'^2 (b - \omega_k) \right) \leq C_0 h^{1/3} L^{2/3}.$$

Using Theorem 2.1 we can rewrite each mode $e(x, y, \eta, \omega_k) = \frac{q(\eta)^{1/6} e^{-iy\eta}}{\sqrt{L'(\omega_k)}} G(x, y, \eta, \omega_k)$ (and hence each \tilde{e}) defined in (4.5) as an integral with phase functions ψ and ζ ; introducing this in (4.16) and applying the Cauchy-Schwartz inequality, the dispersive inequalities for t small reduce to estimates like (4.17) and (4.18), (4.19): using the above Lemma yields

$$\|\mathcal{P}_{h,a}(t, x, y)\|_{L^\infty} \leq h^{-d} h^{1/3} \left(L^{1/3} + h^{1/3} L^{2/3} + h^{2/3} L \right),$$

where we have first obtained a factor $h^{-2/3} = h^{-1} h^{1/3}$ from $q(\theta/h)^{1/3}$ and where the terms in the bracket correspond to (4.17) and (4.18), (4.19). Let now $L = h^{-\epsilon}$ and $t \leq \frac{1}{L}$, then we obtain the bound

$$\|\mathcal{P}_{h,a}(t, x, y)\|_{L^\infty} \leq h^{-d} \left(\left(\frac{h}{t} \right)^{1/3} + \left(\frac{h}{t} \right)^{2/3} + \left(\frac{h}{t} \right) \right) \lesssim h^{-d} \left(\frac{h}{t} \right)^{1/3}.$$

- When the time is larger, $h^{1/3} \ll h^\epsilon \lesssim t$, for values of $k \in [1, h^{-\epsilon}]$ we can apply the stationary phase method for the phase function

$$\psi(x, y, \theta, \omega_k) - \psi(a, 0, \theta, \omega_k) + t\rho(h^{2/3}\omega_k, \theta)$$

and where the symbol is given by

$$(4.20) \quad \check{p}_{0,h} \left(p_0 A i + x p_1 h^{1/3} |\theta|^{-1/3} A i' \right) (\zeta(x, y, \theta/h, \omega_k)) \left(p_0 A i + a p_1 h^{1/3} |\theta|^{-1/3} A i' \right) (\zeta(a, 0, \theta/h, \omega_k)).$$

Indeed, using Theorem 2.1 we can write G in terms of ψ and ζ , where ψ is homogeneous of degree 1 in θ ; from Proposition 5.3 and the homogeneity property of ψ , it follows that it has the following form:

$$\psi(x, y, \theta, \omega) = y\theta + |\theta| \left(B_0(y, \theta/|\theta|) + (|\theta| - 1) B_2(y, \theta/|\theta|) + \mathcal{H}_{j \geq 3} \right).$$

Since $B_0(0, \omega) = 0$ and $B_2(0, \omega) = 0$, we can write

$$\begin{aligned} t\rho(h^{2/3}\omega_k, \theta) + \psi(x, y, \theta, \omega_k) - \psi(a, 0, \theta, \omega_k) &= t\rho(h^{2/3}\omega_k, \theta) + y\theta \\ &\quad + |\theta| (B_0(y, \theta/|\theta|) + (|\theta| - 1) B_2(y, \theta/|\theta|) + \mathcal{H}_{j \geq 3}). \end{aligned}$$

Since $\rho(h^{2/3}\omega_k, \theta) = \sqrt{|\theta|^2 + h^{2/3}\omega_k q^{2/3}(\theta)} \simeq |\theta| + \frac{1}{2}h^{2/3}\omega_k q(\theta/|\theta|)^{2/3}|\theta|^{1/3} + O(h^{4/3}\omega_k^2)$, $B_0(y, \omega) = O(y^3)$ and $B_2(y, \omega) = O(y^2)$, the critical point satisfies in the two dimensional case (when $\theta = |\theta|$) satisfy

$$y(1 + O(y)) + t(1 + \frac{1}{6}h^{2/3}\omega_k\theta^{-2/3} + O((h^{2/3}\omega_k\theta^{-1/3})^2)) = 0.$$

The second derivative of the phase function (with respect to θ) equals $-\frac{t}{9}h^{2/3}\omega_k\theta^{-5/3}(1 + O((h^{2/3}\omega_k)^{1/2}))$: indeed, the terms belonging to $\mathcal{H}_{j \geq 3}$ contain small factors of size $(h^{2/3}\omega_k)^{3/2} \simeq hk$, since if $x > Ch^{2/3}\omega_k$ or $a > Ch^{2/3}\omega_k$ for some constant $C > 2$ then the terms involving Airy functions decrease exponentially. It remains to show that we can apply the stationary phase with large parameter $\frac{t\omega_k}{h^{1/3}}$ for $t > h^\epsilon$, $\epsilon > 0$ like before.

If we consider the second derivative with respect to θ of the symbol (4.20), the main contribution is given by $\omega_k^{11/4}$ and occurs when $\zeta(x, y, \theta/\theta, \omega_k)$ is large and $\zeta(a, 0, \theta/h, \omega_k)$ is bounded and when in the product $Ai(\zeta(x, y, \theta/\theta, \omega_k))Ai(\zeta(a, 0, \theta/h, \omega_k))$ we take the second derivative of the first Airy factor only. Therefore, it suffices to check that there exists $\epsilon > 0$ such that

$$\frac{h^{1/3}}{t\omega_k}\omega_k^{11/4} \ll 1 \quad \text{for } t > h^\epsilon \text{ and } k \lesssim h^{-\epsilon}.$$

This does hold as long as $\epsilon < \frac{2}{13}$. We then estimate, for $\epsilon < 2/13$ and $t > h^\epsilon$:

$$\|\mathcal{P}_{h,a}(t, x, y)\|_{L^\infty} \leq h^{-d+1/3} \sum_{1 \leq k \lesssim h^{-\epsilon}} \frac{1}{k^{1/3}} \left(\frac{h^{1/3}}{t\omega_k}\right)^{1/2} \simeq h^{-d} \left(\frac{h}{t}\right)^{1/2} h^{-\epsilon/3} \lesssim h^{-d} \left(\frac{h}{t}\right)^{1/3}.$$

The last case corresponds to values $2h^{-\epsilon} \leq k \leq 1/h$ and $t \in (0, 1]$, $0 \leq x, a \lesssim h^{\frac{2}{3}(1-\epsilon)}$. For these values it is clear that $|\zeta(x, y, \eta, \omega_k)| \geq \frac{\omega_k}{2}$ and $|\zeta(a, y, \eta, \omega_k)| \geq \frac{\omega_k}{2}$, therefore we can apply the stationary phase method in (4.3) in both s, s' : for small values of $x, a \lesssim h^{\frac{2}{3}(1-\epsilon)}$, we obtain two factors $\simeq h^{1/2}/|\omega_k|^{1/4}$ and since the integration with respect to θ provides a constant independent of h, a , we obtain in this case

$$\|\mathcal{P}_{h,a}(t, x, y)\|_{L^\infty} \leq h^{-d-1/3} \sum_{2h^{-\epsilon} \leq k \leq 1/h} \frac{1}{k^{1/3}} \frac{h}{\omega_k^{1/2}} \lesssim h^{-(d-1)-1/3} \sum_{k \leq 1/h} k^{-2/3} \simeq h^{-d+1/3},$$

which ends the proof of (3.3) in Theorem 3.1, and therefore the proof of (1.1) in Theorem 1.3 is complete. \square

5. APPENDIX

5.1. Some properties of the generating function of χ_M introduced in Lemma 2.2. We present in this appendix the construction of the phase function Γ that generates the canonical transformation χ_M .

Let $\eta = \theta/h$ with $\theta = \varrho\omega$, $\omega = \eta/|\eta|$ for ϱ near 1 and let the glancing set be given by $\mathcal{GL} = \{X = 0, \xi = 0, \varrho - 1 = 0\}$. We will work with formal Taylor expansions near \mathcal{GL} of the form

$$F = \sum_{a,b,c} f_{a,b,c}(Y, \omega) X^a (\varrho - 1)^b \xi^c$$

We attribute a weight to each factor $X, \varrho - 1, \xi$: a monomial of the form $X^a (\varrho - 1)^b \xi^c$ is homogeneous of degree k if and only if $c + 2(a + b) = k$. Given such a formal series $F(X, Y, \xi, \eta)$ defined near the glancing set \mathcal{GL} , we write $F \simeq \sum_{k \geq 0} F_k$, where F_k is homogeneous of degree k ; we also write $F \in \mathcal{H}_{\geq j}$ if and only if $F \simeq \sum_{k \geq j} F_k$, where, like before, F_k is homogeneous of degree k . Therefore,

$$F_0 = f_0(Y, \omega); \quad F_1 = \xi f_1(Y, \omega); \quad F_2 = X f_2^0(Y, \omega) + (\varrho - 1) f_2^1(Y, \omega) + \xi^2 f_2^2(Y, \omega); \quad \text{etc.}$$

Replacing x, Ξ, Θ by their formulas (2.7) (as functions of (X, Y, ξ, η)) and using that from (2.5)

$$(5.1) \quad \Xi^2 + R(X, Y, \Theta) = 1 \quad \text{if and only if} \quad \xi^2 + |\eta|^2 + xq(\eta) = 1,$$

(where notice that y doesn't appear at all) yields

$$(5.2) \quad \begin{cases} B \simeq \sum_{j \geq 0} (\varrho - 1)^j B_{2j}(Y, \omega), \\ A \simeq \sum_{k \geq 1} A_k \end{cases}$$

Indeed, since $\Xi|_{\mathcal{G}} = 0$ we must have $A_0 = 0$. We also have $\Xi(X, Y, \xi, \eta) \in \mathcal{H}_{\geq 1}$ and $x(X, Y, \xi, \eta) \in \mathcal{H}_{\geq 2}$. Moreover, from the proof of Melrose's theorem it follows that if formal series of the form (5.2) satisfy (2.7) and (5.1), then there exist C^∞ functions B, A with the same Taylor development near \mathcal{GL} satisfying (2.7) and (5.1).

We first consider in (5.1) the terms homogeneous of order ≤ 3 . Let $\Xi = \Xi_1 + \Xi_2 + \mathcal{H}_{j \geq 3}$, then $\Xi^2 = \Xi_1^2 + 2\Xi_1\Xi_2 + \mathcal{H}_{j \geq 4}$ and therefore it will be enough to have information on Ξ_1 and Ξ_2 homogeneous of order one and two, respectively. From the third equation in (2.7) this means that it is enough to know A up to second order homogeneity terms. The first equation in (2.7) will allow to obtain x up to $\mathcal{H}_{j \geq 4}$ terms. Let

$$\begin{cases} A = \xi \ell(Y, \omega) + [\alpha(Y, \omega)X + \beta(y, \omega)(\varrho - 1) + \gamma(Y, \omega)\xi^2] + \mathcal{H}_{j \geq 3}, \\ \frac{\partial A}{\partial \xi} = \ell(Y, \omega) + 2\xi\gamma(Y, \omega) + \mathcal{H}_{j \geq 2}, \\ x = X(1 + \ell(Y, \omega)) + 2\xi\gamma(Y, \omega) + X\mathcal{H}_{j \geq 2}. \end{cases}$$

Using $xq(\theta) = 1 - \xi^2 - |\theta|^2$ and that $q(\theta) = \varrho^2 q(\omega)$ yields

$$X(1 + \ell + 2\xi\gamma + \mathcal{H}_{j \geq 2})\varrho^2 q(\omega) = 1 - \xi^2 - \varrho^2$$

which gives the form of X in terms of (Y, ω) , ξ and $\varrho - 1$ (of homogeneity 0, 1, 2, respectively) as follows

$$X = \frac{(1 - \xi^2 - \varrho^2)}{\varrho^2 q(\omega)(1 + \ell)} \left(1 - \frac{2\xi\gamma}{1 + \ell} + \mathcal{H}_{j \geq 2} \right).$$

Eventually we get

$$(5.3) \quad X = \frac{-2(\varrho - 1) - \xi^2}{q(\omega)(1 + \ell)} \left(1 - \frac{2\xi\gamma}{1 + \ell} \right) + \mathcal{H}_{j \geq 4}.$$

We replace (5.3) in $\Xi^2 + R(X, Y, \Theta) = 1$ which gives, using also the third line of (2.7),

$$\Xi = \xi + (\xi\ell + \alpha X + \beta(\varrho - 1) + \gamma\xi^2) + X\alpha = \xi(1 + \ell) + (2\alpha X + \beta(\varrho - 1) + \gamma\xi^2),$$

therefore $\Xi_1 = \xi(1 + \ell)$ and $\Xi_2 = (2\alpha X + \beta(\varrho - 1) + \gamma\xi^2)$ and $\Xi^2 = \xi^2(1 + \ell)^2 + 2(1 + \ell)\xi(2\alpha X + \beta(\varrho - 1) + \gamma\xi^2) + \mathcal{H}_{j \geq 4}$ and using $\Xi^2 + R(X, Y, \Theta) = 1$ yields, modulo $\mathcal{H}_{j \geq 4}$

$$(5.4) \quad \xi^2(1 + \ell)^2 + 2(1 + \ell)\xi(2\alpha X + \beta(\varrho - 1) + \gamma\xi^2) + R_0(Y, \Theta) + XR_1(Y, \Theta) = 1.$$

Here we recall from (2.1) and (2.2) that we had set $R_0 = R|_{x=0} = \Theta^2 + O(Y^2)$ and $R_1 = \frac{\partial R}{\partial x}|_{x=0} = R_1(Y, \Theta)$ (notice that we have $q(\eta) = R_1(0, \eta)$). It remains to develop Θ modulo $\mathcal{H}_{j \geq 4}$. From the forth line of (2.7) we have (see the notations in (5.2)):

$$\Theta = \theta + \frac{\partial B}{\partial Y} + X \frac{\partial A}{\partial Y} = \omega + \frac{\partial B_0}{\partial Y}(Y, \omega) + (\varrho - 1)(\omega + \frac{\partial B_2}{\partial Y}) + X\xi \frac{\partial \ell}{\partial Y} + \mathcal{H}_{j \geq 4},$$

which allows to identify

$$\Theta_0 = \omega + \frac{\partial B_0}{\partial Y}(Y, \omega), \quad \Theta_1 = 0, \quad \Theta_2 = (\varrho - 1)(\omega + \frac{\partial B_2}{\partial Y}), \quad \Theta_3 = X\xi \frac{\partial \ell}{\partial Y}.$$

Now (5.4) becomes, modulo $\mathcal{H}_{j \geq 4}$

$$\xi^2(1 + \ell)^2 + 2(1 + \ell)\xi(2\alpha X + \beta(\varrho - 1) + \gamma\xi^2) + R_0(Y, \Theta_0) + \nabla_{\Theta} R_0(Y, \Theta_0)(\Theta_2 + \Theta_3) + XR_1(Y, \Theta_0) = 1.$$

which reads also as follows

$$(5.5) \quad X \left[R_1(Y, \Theta_0) + 4\alpha\xi(1 + \ell) + \xi \nabla_{\Theta} R_0(Y, \Theta_0) \frac{\partial \ell}{\partial Y} \right] \\ = 1 - R_0(Y, \Theta_0) - \xi^2(1 + \ell)^2 - 2(1 + \ell)\xi(\beta(\varrho - 1) + \gamma\xi^2) - \nabla_{\Theta} R_0(Y, \Theta_0)(\varrho - 1)(\omega + \frac{\partial B_2}{\partial Y}).$$

It remains to introduce (5.3) in the last equation (5.5) and identify the terms homogeneous of degree 0, 1, 2 and 3:

$$(5.6) \quad \begin{cases} R_0(Y, \omega + \frac{\partial B_0}{\partial Y}(Y, \omega)) = 1 \text{ (order 0);} \\ \frac{1}{\frac{q(\omega)(1 + \ell)}{R_1}} = \frac{(1 + \ell)^2}{R_1} \text{ (order 2, terms in } \xi^2); \\ \frac{\nabla_{\Theta} R_0}{R_1}(\omega + \frac{\partial B_2}{\partial Y}) = \frac{2}{(1 + \ell)q(\omega)} \text{ (order 2, terms in } (\varrho - 1)); \\ \left[4\alpha(1 + \ell) + \nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \right] \frac{(1 + \ell)^2}{R_1^2} - \frac{2\gamma(1 + \ell)}{R_1} = \frac{2\gamma}{q(\omega)(1 + \ell)^2} \text{ (order 3, terms in } \xi^3); \\ \left[4\alpha(1 + \ell) + \nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \right] \frac{\nabla_{\Theta} R_0}{R_1^2}(\omega + \frac{\partial B_2}{\partial Y}) - \frac{2\beta(1 + \ell)}{R_1} = \frac{4\gamma}{q(\omega)(1 + \ell)^2} \text{ (order 3, terms in } \xi(\varrho - 1)). \end{cases}$$

Remark 5.1. The system (5.6) consists of five equations, while the unknowns are the following six functions $B_0(Y, \omega)$, $B_2(Y, \omega)$, $\ell(Y, \omega)$, $\alpha(Y, \omega)$, $\beta(Y, \omega)$ and $\gamma(Y, \omega)$. However, the first three equations in (5.6) involve only three unknown functions : B_0 , B_2 and ℓ and we start by determining them.

- Consider the first equation in (5.6):

Lemma 5.2. *For $\theta \in \mathbb{R}^{d-1} \setminus \{0\}$, there exists a unique function $\phi(Y, \theta)$, homogeneous of degree 1 in θ , solving the eikonal equation*

$$R_0(Y, \partial_Y \phi) = \theta^2, \quad \phi|_{Y=0} = 0.$$

Since $R_0(Y, \Theta) = \Theta^2 + O(Y^2)$, we must have $\phi(Y, \theta) = Y\theta(1 + O(Y^2))$. It suffices to take

$$(5.7) \quad B_0(Y, \omega) = \phi(Y, \omega) - Y\omega.$$

As a consequence we have $\nabla_Y^2 B_0(0, \omega) = 0$, $\nabla_Y B_0(0, \omega) = 0$ and $B_0(0, \omega) = 0$.

- Consider the next two equations in (5.6), corresponding to terms homogeneous of order 2: the second equation defines the function ℓ since we have:

$$(1 + \ell)^3 = \frac{R_1(Y, \omega + \frac{\partial B_0}{\partial Y})}{q(\omega)},$$

and using the form of B_0 and that $q(\omega) = R_1(0, \omega)$ we also find $\ell(0, \omega) = 0$. It then follows that the third equation in (5.6) is a linear transport equation for $B_2(Y, \omega)$ and we can take $B_2|_{Y=0} = 0$ (at $Y = 0$, the transport field is $2\omega \nabla_Y$).

- Consider now the last two equations in (5.6), which form a linear system in α, β, γ .

$$(5.8) \quad \begin{cases} \frac{\alpha}{q(1+\ell)} - \gamma = -\nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \frac{(1+\ell)}{4R_1}, \\ \frac{\alpha}{q(1+\ell)} - \frac{\beta}{4} - \frac{\gamma}{2} = -\nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \frac{(1+\ell)}{4R_1}, \end{cases}$$

which gives $\gamma = 2\beta$ and we can take $\alpha = 0$. Moreover, at $Y = 0$, we have

$$3\nabla_Y \ell(0, \omega) = \frac{1}{q(\omega)} \nabla_Y R_1(0, \omega),$$

therefore it follows that

$$\nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \frac{(1+\ell)}{R_1} \Big|_{Y=0} = \frac{1}{3R_1^2(0, \omega)} \{R_0, R_1\}(0, \omega) = \beta(0, \omega).$$

We have obtained the following:

Proposition 5.3. *The phase function Γ has the following form near the glancing set \mathcal{G} :*

$$\begin{cases} \Gamma(X, Y, \xi, \theta) = B(Y, \omega) + X \left[\xi \ell(Y, \omega) + \gamma(Y, \omega)(\xi^2 + \theta^2 - 1) + \mathcal{H}_{j \geq 3} \right], \\ B(Y, \omega) = B_0(Y, \omega) + (\varrho - 1)B_2(Y, \omega) + \mathcal{H}_{j \geq 3}, \end{cases}$$

where $\omega = \theta/|\theta|$, $\varrho = |\theta|$. Also, B_0 is defined in (5.7), B_2 and ℓ are described above, while γ is given by

$$\gamma(Y, \omega) = \nabla_{\Theta} R_0 \frac{\partial \ell}{\partial Y} \frac{(1+\ell)}{4R_1}.$$

Moreover, we have

$$\chi_M \left(\mathcal{GL}_M = \{x = 0, \xi = 0, \theta^2 = 1\} \right) = \mathcal{GL} = \{X = 0, \Xi = 0, R_0(Y, \Theta) = 1\}.$$

The restriction to the glancing regime is given by

$$y = Y + \frac{\partial B}{\partial \theta}(Y, \theta)|_{\varrho=1}, \quad \Theta = \theta + \frac{\partial B_0}{\partial Y}(Y, \theta).$$

Remark 5.4. The restriction of the canonical transformation χ_M to the glancing set \mathcal{GL}_M preserves the canonical foliation: $\chi_M\left(\{y = y_0 + s\theta_0, \theta = \omega_0, \omega_0^2 = 1\}\right)$ is an integral curve of H_{R_0} on $R_0 = 1$. Be aware that the natural parametrization given by the symbols of the operators is not preserved.

To end the proof of lemma 2.2, it remains to find in (5.1) the terms homogeneous of order > 3 . It turns out that this gives a cascade of linear equations (of the same nature than (5.8) for the determination of α, β, γ) which can be easily solved by induction. Observe that the fact the formal expansion is not uniquely defined reflect the fact that the group of canonical transformation which preserves the model $\{x = 0, \xi^2 + |\eta|^2 + xq(\eta) = 1\}$ is not trivial.

REFERENCES

- [1] K. G. Andersson and R. B. Melrose. The propagation of singularities along gliding rays. *Invent. Math.*, 41(3):197–232, 1977.
- [2] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. Strichartz estimates for the wave equation on manifolds with boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1817–1829, 2009.
- [3] Gregory Eskin. Parametrix and propagation of singularities for the interior mixed hyperbolic problem. *J. Analyse Math.*, 32:17–62, 1977.
- [4] Lars Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].
- [5] Oana Ivanovici. Counterexamples to Strichartz estimates for the wave equation in domains. *Math. Ann.*, 347(3):627–673, 2010.
- [6] Oana Ivanovici. Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary. *J. Eur. Math. Soc. (JEMS)*, 14(5):1357–1388, 2012.
- [7] Oana Ivanovici and Gilles Lebeau. Dispersion for the wave and Schrödinger equation outside a strictly convex obstacle. preprint, 2016.
- [8] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case. *Ann. of Math. (2)*, 180(1):323–380, 2014.
- [9] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Estimations de strichartz pour les ondes dans le modèle de friedlander en dimension 3. In *Séminaire Laurent Schwartz – EDP et applications, 2013–2014*, Sémin. Équ. Dériv. Partielles, pages Exp. No. III, 12. École Polytech., Palaiseau, 2014.
- [10] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Strichartz estimates for the wave equation on a 2d model convex domain. preprint, 2016.
- [11] V. Ja. Ivrii. Propagation of the singularities of a solution of the wave equation near the boundary. *Dokl. Akad. Nauk SSSR*, 239(4):772–774, 1978.
- [12] R. B. Melrose. Equivalence of glancing hypersurfaces. *Invent. Math.*, 37(3):165–191, 1976.
- [13] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.
- [14] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2):129–168, 1982.
- [15] Richard B. Melrose and Michael E. Taylor. *Boundary Problems for Wave Equations With Grazing and Gliding Rays*. Available at <http://www.unc.edu/math/Faculty/met/wavep.html>.

- [16] Richard B. Melrose and Michael E. Taylor. Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Adv. in Math.*, 55(3):242–315, 1985.
- [17] Richard B. Melrose and Michael E. Taylor. The radiation pattern of a diffracted wave near the shadow boundary. *Comm. Partial Differential Equations*, 11(6):599–672, 1986.
- [18] Hart F. Smith and Christopher D. Sogge. On the critical semilinear wave equation outside convex obstacles. *J. Amer. Math. Soc.*, 8(4):879–916, 1995.
- [19] Hart F. Smith and Christopher D. Sogge. On the L^p norm of spectral clusters for compact manifolds with boundary. *Acta Math.*, 198(1):107–153, 2007.
- [20] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.
- [21] Maciej Zworski. High frequency scattering by a convex obstacle. *Duke Math. J.*, 61(2):545–634, 1990.

UNIVERSITÉ NICE SOPHIA-ANTIPOLIS, LABORATOIRE JAD, UMR CNRS 7351, PARC VALROSE,
06108 NICE CEDEX 02, FRANCE

E-mail address: oana.ivanovici@unice.fr

UNIVERSITÉ PARIS DIDEROT, INSTITUT DE MATHÉMATIQUES DE JUSSIEU PARIS RIVE GAUCHE,
UMR CNRS 7586, UPMC - CAMPUS JUSSIEU, 4, PLACE JUSSIEU, BOITE COURRIER 247 - 75252
PARIS CEDEX 05

E-mail address: richard.lascar@imj-prg.fr

UNIVERSITÉ NICE SOPHIA-ANTIPOLIS, LABORATOIRE JAD, UMR CNRS 7351, PARC VALROSE,
06108 NICE CEDEX 02, FRANCE

E-mail address: gilles.lebeau@unice.fr

UNIVERSITÉ NICE SOPHIA-ANTIPOLIS, LABORATOIRE JAD, UMR CNRS 7351, PARC VALROSE,
06108 NICE CEDEX 02, FRANCE AND INSTITUT UNIVERSITAIRE DE FRANCE

E-mail address: fabrice.planchon@unice.fr